

APPROXIMATE INNERNESS AND CENTRAL TRIVIALITY OF ENDOMORPHISMS

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ABSTRACT. We introduce the notions of approximate innerness and central triviality for endomorphisms on separable von Neumann factors, and we characterize them for hyperfinite factors by Connes-Takesaki modules of endomorphisms and modular endomorphisms which are introduced by Izumi. Our result is a generalization of the corresponding result obtained by Kawahigashi-Sutherland-Takesaki in automorphism case.

1. INTRODUCTION

The purpose of this paper is first to introduce two notions, “approximate innerness” and “central triviality” for endomorphisms on factors, and second to generalize the result of Y. Kawahigashi, C. E. Sutherland and M. Takesaki [19] to endomorphism case.

The study of automorphisms or group actions has drawn attentions in studies of operator algebras. From the viewpoint of the classification theory of group actions, two classes of automorphisms have been considered significant, i.e., approximately inner automorphisms $\overline{\text{Int}}(M)$ and centrally trivial automorphisms $\text{Cnt}(M)$ on a factor M , which are studied by A. Connes [4, 6]. In particular, A. Ocneanu obtained the uniqueness result for approximately inner and centrally free actions of discrete amenable groups on McDuff factors [28].

On those two properties, Connes announced the following characterization without a proof, using the flow of weights [5]: for any hyperfinite factor M ,

- (1) $\overline{\text{Int}}(M) = \text{Ker}(\text{mod})$,
- (2) $\text{Cnt}(M) = \{\text{Ad } u \cdot \sigma_c^\varphi \mid u \in U(M), c \in Z^1(F^M, \mathbb{C})\}$,

where $\text{mod}: \text{Aut}(M) \rightarrow \text{Aut}(F^M)$ is the Connes-Takesaki module map [7], $Z^1(F^M, \mathbb{C})$ is the set of scalar valued 1-cocycles for the flow of weights F^M and $U(M)$ is the set of all unitary elements in M .

A proof of this theorem was first presented by Kawahigashi, Sutherland and Takesaki [19]. Their result well motivates us to consider a generalization to endomorphisms, but our work also relies on the study of actions because an action of a compact group dual (or more generally a discrete quantum group) is essentially identical to a Roberts action [30]. Indeed, using the present work, we will obtain a uniqueness result for approximately inner and centrally free actions of an amenable discrete Kac algebra on hyperfinite type III factors [25].

2000 *Mathematics Subject Classification.* Primary 46L10; Secondary 46L37.

^{1,2,3} Supported by JSPS.

Now let us explain the details of this paper. In the first half, we study approximately inner endomorphisms on a factor M . We introduce a topology on the set of endomorphisms with finite index, and then discuss approximation by inner endomorphisms. Hence it is convenient to introduce the notion of rank for an approximately inner endomorphism, that is, if ρ is approximated by inner endomorphisms with dimension r , then we will say that ρ has rank r . In fact, we can define the rank for any positive real number so that we match approximate innerness to the theory of the Connes-Takesaki modules for endomorphisms introduced by M. Izumi [17]. Then we study the set of approximately inner endomorphisms with rank $r > 0$, which we denote by $\overline{\text{Int}}_r(M)$.

In the latter half, we study the set of centrally trivial endomorphisms on a factor M which we denote by $\text{Cnd}(M)$. Our aim is to clarify the relation between $\text{Cnd}(M)$ and $\text{End}(M)_m$, the set of modular endomorphisms introduced by Izumi [17]. Our main result is the following (Theorem 3.15, 4.12):

Main Theorem. *Let M be a hyperfinite factor. Then one has the following:*

- (1) $\overline{\text{Int}}_r(M) = \{\rho \in \text{End}(M)_{\text{CT}} \mid \text{mod}(\rho) = \theta_{\log(r/d(\rho))}\}$ for any $r > 0$.
- (2) $\text{Cnd}(M) = \text{End}(M)_m$.

Here, $\text{End}(M)_{\text{CT}}$ is the set of endomorphisms on M which have Connes-Takesaki modules.

We should emphasize that we make use of the main result of [19] and their idea on discrete decompositions, but we mainly use Popa's theory on approximate innerness and central freeness of subfactors [29]. Indeed, some discussions of [19] involve the classification results of discrete group actions [28, 31], and it does not seem that those are applicable to the endomorphism case at ease.

2. APPROXIMATE INNERNESS OF ENDOMORPHISMS

First we fix notations. In this paper, we treat only von Neumann algebras with separable preduals except for ultraproduct von Neumann algebras. Let M be a von Neumann algebra. For $\varphi \in M_*$ and $a \in M$, we define the functionals φa and $a\varphi$ in M_* by $\varphi a(x) := \varphi(ax)$ and $a\varphi(x) := \varphi(xa)$ for all $x \in M$. We denote by $U(M)$ the set of all unitary elements in M . We denote by $W(M)$ and $W_{\text{lac}}(M)$ the sets of faithful normal semifinite weights and faithful normal semifinite lacunary weights on M , respectively [2, 34]. For a faithful state $\phi \in M_*$, we set $|x|_\phi := \phi(|x|)$ and $\|x\|_\phi := \phi(x^*x)^{1/2}$. Note that $|\cdot|_\phi$ satisfies the triangle inequality on the centralizer of ϕ . We denote by M_ϕ the centralizer of ϕ , that is, $x \in M_\phi$ if and only if $\phi(xy) = \phi(yx)$ for any $y \in M$.

Let $\mathcal{H} \subset M$ be a subspace. We say that \mathcal{H} is a Hilbert space in M if $\mathcal{H} \subset M$ is σ -weakly closed and $\eta^*\xi \in \mathbb{C}$ for all $\xi, \eta \in \mathcal{H}$ [30]. The smallest projection $e \in M$ such that $e\mathcal{H} = \mathcal{H}$ is called the support of \mathcal{H} .

We denote by $\text{End}(M)$ and $\text{Sect}(M)$ the set of normal endomorphisms and sectors on M , that is, $\text{Sect}(M)$ is the set of equivalence classes of endomorphisms on M by unitary equivalence. For two endomorphisms $\rho, \sigma \in \text{End}(M)$, we let $(\rho, \sigma) = \{v \in M \mid v\rho(x) = \sigma(x)v \text{ for all } x \in M\}$. If ρ is irreducible, then (ρ, σ)

is a Hilbert space with the inner product $(V, W) = W^*V$ for $V, W \in (\rho, \sigma)$. For an endomorphism ρ on M , a left inverse ϕ of ρ means a faithful normal unital completely positive map on M with $\phi \circ \rho = \text{id}$. For a factor M , we denote by $\text{End}(M)_0$ the set of endomorphisms with finite index. For $\rho \in \text{End}(M)_0$, E_ρ denotes the minimal expectation from M onto $\rho(M)$ [14]. We define the standard left inverse of ρ by $\phi_\rho = \rho^{-1} \circ E_\rho$.

2.1. A topology on the set of endomorphisms

We define the topology on the set of endomorphisms on a factor, which is related to [24, Definition 3.1].

Definition 2.1. Let N be a factor. We introduce the topology \mathcal{T}_ϕ on $\text{End}(N)_0$ by giving the following neighborhoods at $\rho_0 \in \text{End}(N)_0$ which are defined for $n \in \mathbb{N}$, a finite family $\{\varphi_i\}_{i=1}^n \subset N_*$ and $\varepsilon > 0$ by

$$U(\rho_0; \varphi_1, \dots, \varphi_n, \varepsilon) = \{\rho \in \text{End}(N)_0 \mid \|\varphi_i \circ \phi_\rho - \varphi_i \circ \phi_{\rho_0}\| < \varepsilon \text{ for all } i = 1, \dots, n\}.$$

The topology \mathcal{T}_ϕ is metrizable. Take a norm dense sequence $\{\varphi_n\}_{n=1}^\infty$ in the set of normal states on N_* . The following metric d_ϕ defines the topology \mathcal{T}_ϕ ,

$$d_\phi(\rho, \sigma) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|\varphi_n \circ \phi_\rho - \varphi_n \circ \phi_\sigma\| \text{ for } \rho, \sigma \in \text{End}(N)_0.$$

The restriction of \mathcal{T}_ϕ on $\text{Aut}(N)$ coincides with the u -topology [11, Definition 3.4] as seen below.

Lemma 2.2. Let α^ν , $\nu \in \mathbb{N}$, and α be in $\text{Aut}(N)$. Then $\alpha^\nu \rightarrow \alpha$ as $\nu \rightarrow \infty$ in the topology \mathcal{T}_ϕ if and only if $\alpha^\nu \rightarrow \alpha$ as $\nu \rightarrow \infty$ in the u -topology.

Proof. The if part is trivial. We show the only if part as follows. The sequence $\{\alpha^\nu\}_\nu$ converges to α as $\nu \rightarrow \infty$ in the topology \mathcal{T}_ϕ if and only if $\|\varphi \circ (\alpha^\nu)^{-1} - \varphi \circ \alpha^{-1}\| \rightarrow 0$ as $\nu \rightarrow \infty$ for all $\varphi \in N_*$. If we put $\psi \circ \alpha$ ($\psi \in N_*$) for φ , we obtain $\|\psi \circ \alpha \circ (\alpha^\nu)^{-1} - \psi\| = \|\psi \circ \alpha - \psi \circ \alpha^\nu\|$. Hence $\|\varphi \circ (\alpha^\nu)^{-1} - \varphi \circ \alpha^{-1}\| \rightarrow 0$ as $\nu \rightarrow \infty$ for all $\varphi \in N_*$ if and only if $\|\psi \circ \alpha^\nu - \psi \circ \alpha\| = 0$ as $\nu \rightarrow \infty$ for all $\psi \in N_*$, which means the convergence of $\{\alpha^\nu\}_\nu$ to α in the u -topology. \square

Note that the u -topology on $\text{Aut}(N)$ is metrizable and complete. By the previous lemma, the restriction of the metric d_ϕ on $\text{Aut}(N)$ gives the u -topology, but $\text{Aut}(N) \subset \text{End}(N)_0$ is not closed in general. Namely, that restriction may not be a complete metric.

2.2. Approximate innerness

Let $\mathcal{H} \subset N$ be a finite dimensional Hilbert space with support 1, and $\{v_i\}_i \subset \mathcal{H}$ an orthonormal basis. Then $\rho_{\mathcal{H}}(x) := \sum_i v_i x v_i^*$ gives an endomorphism of N . We say that $\rho \in \text{End}(N)$ is *inner* if there exists a Hilbert space $\mathcal{H} \subset N$ such that $\rho = \rho_{\mathcal{H}}$. Then we have $d(\rho) = \dim(\mathcal{H})$. We also say that ρ has *rank* $\dim(\mathcal{H})$. We denote by $\text{Int}_d(N)$ the set of inner endomorphisms of rank d . Now we introduce approximate innerness of endomorphisms.

Definition 2.3. Let $\rho \in \text{End}(N)_0$ and $r \in \mathbb{N}$. We say that ρ is an *approximately inner endomorphism of rank r* if for each $\nu \in \mathbb{N}$, there exists an r -dimensional Hilbert space $\mathcal{H}^\nu \subset N$ with support 1 such that $(\rho_{\mathcal{H}^\nu})_\nu$ converges to ρ with respect to the topology \mathcal{T}_ϕ .

We generalize this notion for general $r > 0$ as follows.

Definition 2.4. Let N be a factor. Let $\rho \in \text{End}(N)$ and $r > 0$. We say that ρ is an *approximately inner endomorphism of rank r* if there exist sequences of partial isometries $\{v_i^\nu\}_{i=1}^{[r]+1} \subset N$, $\nu \in \mathbb{N}$, such that

- (1) $(v_i^\nu)^* v_i^\nu = 1$ for $1 \leq i \leq [r]$ for all $\nu \in \mathbb{N}$,
- (2) if $r \in \mathbb{N}$, $v_{[r]+1}^\nu = 0$ for all $\nu \in \mathbb{N}$,
- (3) $\sum_{i=1}^{[r]+1} v_i^\nu (v_i^\nu)^* = 1$ for all $\nu \in \mathbb{N}$,
- (4) $\lim_{\nu \rightarrow \infty} \left\| \frac{1}{r} v_i^\nu \varphi - (\varphi \circ \phi_\rho) v_i^\nu \right\| = 0$ for all $\varphi \in N_*$.

We denote by $\overline{\text{Int}}_r(N)$ the set of approximately inner endomorphisms of rank r . By definition, we have $\overline{\text{Int}}(N) = \text{Aut}(M) \cap \overline{\text{Int}}_1(N)$.

2.3. Locally trivial subfactors

We recall locally trivial subfactors introduced in [29, Chapter 2]. Let P be a factor. For $\rho_0 := \text{id}$ and $\rho_1 := \rho \in \text{End}(P)_0$, the locally trivial subfactor $N^{(\text{id}, \rho)} \subset M^{(\text{id}, \rho)}$ is defined as follows:

$$M^{(\text{id}, \rho)} = P \otimes M_2(\mathbb{C}),$$

$$N^{(\text{id}, \rho)} = \{x \otimes e_{00} + \rho(x) \otimes e_{11} \mid x \in P\},$$

where $\{e_{ij}\}_{i,j=0}^1$ denotes a system of matrix units of $M_2(\mathbb{C})$. The canonical isomorphism from P onto $N^{(\text{id}, \rho)}$ is denoted by $\alpha^{(\text{id}, \rho)}$.

For $\mu_0, \mu_1 > 0$ with $\mu_0 + \mu_1 = 1$, we set $\mu := (\mu_0, \mu_1)$. We define the unital completely positive map $\phi_\rho^\mu: M^{(\text{id}, \rho)} \rightarrow P$ by

$$\phi_\rho^\mu(x) = \sum_{i=0}^1 \mu_i \phi_{\rho_i}(x_{ii}), \quad x \in M^{(\text{id}, \rho)}.$$

Then ϕ_ρ^μ has the following property:

$$\phi_\rho^\mu(\alpha^{(\text{id}, \rho)}(a) x \alpha^{(\text{id}, \rho)}(b)) = a \phi_\rho^\mu(x) b.$$

This implies the map $E^\mu := \alpha^{(\text{id}, \rho)} \circ \phi_\rho^\mu$ is a faithful normal conditional expectation from $M^{(\text{id}, \rho)}$ onto $N^{(\text{id}, \rho)}$. By using the local index formula [20, Theorem 4.4], we have the following.

Lemma 2.5. *One has $\text{Ind}(E^\mu) = \mu_0^{-1} + \mu_1^{-1} d(\rho_1)^2 = \mu_0^{-1} + \mu_1^{-1} \text{Ind}(E_\rho)$.*

2.4. Ultraproduct von Neumann algebras and Central sequence inclusions

We recall the notion of ultraproduct von Neumann algebras and central sequence inclusions. Our standard references are [24, 28, 29].

Let M be a von Neumann algebra and ω a free ultrafilter on \mathbb{N} . Denote by $\mathcal{T}_\omega(M) \subset \ell^\infty(\mathbb{N}, M)$ the C^* -subalgebra which consists of sequences ω -converging to 0 in the strong* topology. Let $\mathcal{N}(\mathcal{T}_\omega(M))$ be the C^* -subalgebra of $\ell^\infty(\mathbb{N}, M)$ normalizing $\mathcal{T}_\omega(M)$. Then the quotient C^* -algebra $M^\omega := \mathcal{N}(\mathcal{T}_\omega(M))/\mathcal{T}_\omega(M)$ has a predual and hence is a von Neumann algebra. We call M^ω the ultraproduct von Neumann algebra of M . The quotient map is denoted by π_ω . We say that $(x^\nu)_\nu \in \ell^\infty(\mathbb{N}, M)$ is a representing sequence of $x \in M^\omega$ if $x = \pi_\omega((x^\nu)_\nu)$. We denote by τ^ω the canonical faithful normal conditional expectation from M^ω onto M , that is, for $x = \pi_\omega((x^\nu)_\nu) \in M^\omega$, we have $\tau^\omega(x) = \lim_{\nu \rightarrow \omega} x^\nu$, where the ultralimit is taken with respect to the σ -weak topology of M . For $\varphi \in M_*$, we define the normal functional $\varphi^\omega \in (M^\omega)_*$ by $\varphi^\omega(x) := \varphi(\tau^\omega(x))$ for $x \in M^\omega$.

Next we consider an inclusion $N \xrightarrow{E} M$, where E is a faithful normal conditional expectation from M onto N . We define the following C^* -subalgebras in $\ell^\infty(\mathbb{N}, M)$:

$$M_\omega^0(E) = \{(x^\nu)_\nu \in \ell^\infty(\mathbb{N}, M) \mid \lim_{\nu \rightarrow \omega} \|\varphi \circ E, x^\nu\| = 0 \text{ for all } \varphi \in N_*\},$$

$$N_\omega^0(E) = \{(x^\nu)_\nu \in \ell^\infty(\mathbb{N}, N) \mid \lim_{\nu \rightarrow \omega} \|\varphi, x^\nu\| = 0 \text{ for all } \varphi \in N_*\}.$$

Then we have an inclusion $N_\omega^0(E) \subset M_\omega^0(E)$ with the conditional expectation $E_\omega^0: M_\omega^0(E) \rightarrow N_\omega^0(E)$ defined by $E_\omega^0((x^\nu)_\nu) = (E(x^\nu))_\nu$. We define the central sequence von Neumann algebras $M_\omega(E)$ and $N_\omega(E)$ by

$$M_\omega(E) := M_\omega^0(E)/\mathcal{T}_\omega(M), \quad N_\omega(E) := (N_\omega^0(E) + \mathcal{T}_\omega(M))/\mathcal{T}_\omega(M).$$

Since E_ω^0 preserves $\mathcal{T}_\omega(M)$, that is, $E_\omega^0(\mathcal{T}_\omega(M)) \subset \mathcal{T}_\omega(M)$, we can naturally define the conditional expectation $E_\omega: M_\omega(E) \rightarrow N_\omega(E)$, which is faithful and normal.

The inclusion $N_\omega(E) \xrightarrow{E_\omega} M_\omega(E)$ is called the central sequence inclusion of $N \xrightarrow{E} M$. Note that $M_\omega(E)$ is finite. Indeed, a functional $\varphi^\omega \circ E_\omega$ is a faithful normal tracial state for all faithful state $\varphi \in N_*$.

If $N = M$, we denote by M_ω for $M_\omega(\text{id}_M)$. Elements in $M_\omega^0(\text{id}_M)$ are said to be ω -centralizing.

Now we consider central sequence inclusions arising from locally trivial subfactors. Let P be a factor and $\rho \in \text{End}(P)_0$. For each $\mu_0, \mu_1 > 0$ with $\mu_0 + \mu_1 = 1$, the following locally trivial subfactor is defined as in the previous subsection:

$$N^{(\text{id}, \rho)} \xrightarrow{E^\mu} M^{(\text{id}, \rho)}.$$

Consider its central sequence inclusion

$$N_\omega^{(\text{id}, \rho)}(E^\mu) \xrightarrow{E_\omega^\mu} M_\omega^{(\text{id}, \rho)}(E^\mu).$$

Note that $M_\omega^{(\text{id}, \rho)}(E^\mu)$ is a subalgebra of $P^\omega \otimes M_2(\mathbb{C})$.

Lemma 2.6. Set $x := \sum_{i,j=0}^1 x_{ij} \otimes e_{ij} \in P^\omega \otimes M_2(\mathbb{C})$. Then $x \in M_\omega^{(\text{id}, \rho)}(E^\mu)$ if and only if for each $0 \leq i, j \leq 1$, a representing sequence $(x_{ij}^\nu)_\nu$ of x_{ij} satisfies

$$\lim_{\nu \rightarrow \omega} \|\mu_i(\varphi \circ \phi_{\rho_i})x_{ij}^\nu - \mu_j x_{ij}^\nu(\varphi \circ \phi_{\rho_j})\| = 0 \quad \text{for all } \varphi \in P_*.$$

Proof. Let $\varphi \in P_*$. Then for $y = \sum_{i,j=0}^1 y_{ij} \otimes e_{ij}$, we have

$$\varphi \circ (\alpha^{(\text{id}, \rho)})^{-1} \circ E^\mu((x_{ij}^\nu \otimes e_{ij})y) = \mu_i \varphi(\phi_{\rho_i}(x_{ij}^\nu y_{ji})),$$

and

$$\varphi \circ (\alpha^{(\text{id}, \rho)})^{-1} \circ E^\mu(y(x_{ij}^\nu \otimes e_{ij})) = \mu_j \varphi(\phi_{\rho_j}(y_{ji} x_{ij}^\nu)).$$

Setting $x^\nu := \sum_{i,j=0}^1 x_{ij}^\nu \otimes e_{ij}$, we have

$$\begin{aligned} [\varphi \circ (\alpha^{(\text{id}, \rho)})^{-1} \circ E^\mu, x^\nu](y) &= \sum_{i,j=0}^1 \mu_i \varphi(\phi_{\rho_i}(x_{ij}^\nu y_{ji})) - \mu_j \varphi(\phi_{\rho_j}(y_{ji} x_{ij}^\nu)) \\ &= \sum_{i,j=0}^1 (\mu_i(\varphi \circ \phi_{\rho_i})x_{ij}^\nu - \mu_j x_{ij}^\nu(\varphi \circ \phi_{\rho_j}))(y_{ji}). \end{aligned}$$

This implies the following inequalities:

$$\|[\varphi \circ (\alpha^{(\text{id}, \rho)})^{-1} \circ E^\mu, x^\nu]\| \leq \sum_{i,j=0}^1 \|\mu_i(\varphi \circ \phi_{\rho_i})x_{ij}^\nu - \mu_j x_{ij}^\nu(\varphi \circ \phi_{\rho_j})\|$$

and

$$\|\mu_i(\varphi \circ \phi_{\rho_i})x_{ij}^\nu - \mu_j x_{ij}^\nu(\varphi \circ \phi_{\rho_j})\| \leq \|[\varphi \circ (\alpha^{(\text{id}, \rho)})^{-1} \circ E^\mu, x^\nu]\|.$$

Therefore $x \in M_\omega^{(\text{id}, \rho)}(E)$ if and only if $\|\mu_i(\varphi \circ \phi_{\rho_i})x_{ij}^\nu - \mu_j x_{ij}^\nu(\varphi \circ \phi_{\rho_j})\| \rightarrow 0$ as $\nu \rightarrow \omega$ for all $0 \leq i, j \leq 1$. \square

The previous lemma implies that the projection $1 \otimes e_{ii}$ is in the relative commutant $(N_\omega^{(\text{id}, \rho)}(E^\mu))' \cap M_\omega^{(\text{id}, \rho)}(E^\mu)$. Then we see that $x \otimes e_{ii} \in P^\omega \otimes \mathbb{C}e_{ii}$ is contained in $(1 \otimes e_{ii})M_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{ii})$ if and only if $\|[\varphi \circ \phi_{\rho_i}, x^\nu]\| \rightarrow 0$ as $\nu \rightarrow \omega$ for all $\varphi \in P_*$. This means $\pi_\omega((x^\nu)_\nu)$ is contained in $P_\omega(E_{\rho_i})$, where $\rho_i(P_\omega) \stackrel{(E_{\rho_i})_\omega}{\subset} P_\omega(E_{\rho_i})$ is the central sequence inclusion of $\rho_i(P) \stackrel{E_{\rho_i}}{\subset} P$. Summarizing these arguments, we have

$$\begin{aligned} (1 \otimes e_{ii})M_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{ii}) &= P_\omega(E_{\rho_i}) \otimes \mathbb{C}e_{ii}, \\ (1 \otimes e_{ii})N_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{ii}) &= \rho_i(P_\omega) \otimes \mathbb{C}e_{ii}. \end{aligned}$$

Namely, the central sequence inclusion of $\rho_i(P) \stackrel{E_{\rho_i}}{\subset} P$ is isomorphic to the corner of $N_\omega^{(\text{id}, \rho)}(E^\mu) \subset M_\omega^{(\text{id}, \rho)}(E^\mu)$ cut by $1 \otimes e_{ii}$. Hence we have the following lemma.

Lemma 2.7. One has

$$\begin{aligned} (1 \otimes e_{00})M_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{00}) &= P_\omega \otimes \mathbb{C}e_{00}, \\ (1 \otimes e_{00})Z(M_\omega^{(\text{id}, \rho)}(E^\mu))(1 \otimes e_{00}) &= Z(P_\omega) \otimes \mathbb{C}e_{00}. \end{aligned}$$

The following proposition clarifies the relations between approximately inner endomorphisms and central sequence inclusions.

Proposition 2.8. *Let $N^{(\text{id}, \rho)} \stackrel{E^\mu}{\subset} M^{(\text{id}, \rho)}$ as before. Let $\text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)}$ be the center valued trace of $M_\omega^{(\text{id}, \rho)}(E^\mu)$. Then the following statements are equivalent:*

- (1) *The central support of $1 \otimes e_{00}$ in $M_\omega^{(\text{id}, \rho)}(E^\mu)$ is equal to 1.*
- (2) *$Z(N_\omega^{(\text{id}, \rho)}(E^\mu)) = Z(M_\omega^{(\text{id}, \rho)}(E^\mu))$.*
- (3) *$\text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)}(1 \otimes e_{ii}) = \mu_i$ for $i = 0, 1$.*
- (4) *$\rho \in \overline{\text{Int}}_{\mu_1/\mu_0}(P)$.*
- (5) *There exist a finite set I and $x_j = \pi_\omega((x_j^\nu)_\nu) \in P^\omega$, $j \in I$ such that*
 - (a) $\bigvee_{j \in I} s(x_j x_j^*) = 1$,
 - (b) $\lim_{\nu \rightarrow \omega} \|\mu_1(\varphi \circ \phi_\rho) x_j^\nu - \mu_0 x_j^\nu \varphi\| = 0$ for all $\varphi \in P_*$.

Proof. (1) \Rightarrow (2). The inclusion $Z(N_\omega^{(\text{id}, \rho)}(E^\mu)) \subset Z(M_\omega^{(\text{id}, \rho)}(E^\mu))$ always holds [29, Corollary 1.3.7 (i)]. Let $z \in Z(M_\omega^{(\text{id}, \rho)}(E^\mu))$. Since $Z(M_\omega^{(\text{id}, \rho)}(E^\mu))(1 \otimes e_{00}) = Z(P_\omega) \otimes e_{00}$ by the previous lemma, there exists $z_0 \in Z(P_\omega)$ such that $z(1 \otimes e_{00}) = z_0 \otimes e_{00}$. Set $z' = (\alpha^{(\text{id}, \rho)})^\omega(z_0)$, where $(\alpha^{(\text{id}, \rho)})^\omega$ is an embedding $P^\omega \rightarrow M_\omega^{(\text{id}, \rho)}(E^\mu)$ naturally defined through $\alpha^{(\text{id}, \rho)}$. Then $z' \in Z(N_\omega^{(\text{id}, \rho)}(E^\mu))$, and $z' \in Z(M_\omega^{(\text{id}, \rho)}(E^\mu))$. By assumption, $z(1 \otimes e_{00}) = z'(1 \otimes e_{00})$ yields $z = z'$.

(2) \Rightarrow (3). Let $\text{CTr}_{N_\omega^{(\text{id}, \rho)}(E^\mu)}$ and $\text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)}$ be the center valued traces of $N_\omega^{(\text{id}, \rho)}(E^\mu)$ and $M_\omega^{(\text{id}, \rho)}(E^\mu)$, respectively. Then the maps $E_\omega^\mu \circ \text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)}$ and $\text{CTr}_{N_\omega^{(\text{id}, \rho)}(E^\mu)} \circ E_\omega^\mu$ are faithful normal conditional expectations from $M_\omega^{(\text{id}, \rho)}(E^\mu)$ onto $Z(N_\omega^{(\text{id}, \rho)}(E^\mu))$. Since those conditional expectations preserve a faithful normal trace of the form $\varphi^\omega \circ E_\omega^\mu$, $\varphi \in N_*$, we have the equality

$$E_\omega^\mu \circ \text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)} = \text{CTr}_{N_\omega^{(\text{id}, \rho)}(E^\mu)} \circ E_\omega^\mu.$$

Since $\text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)}(1 \otimes e_{ii})$ is contained in $Z(M_\omega^{(\text{id}, \rho)}(E^\mu))$, it is also contained in $Z(N_\omega^{(\text{id}, \rho)}(E^\mu))$ by the assumption of (2). Using $E_\omega^\mu(1 \otimes e_{ii}) = \mu_i$, we have

$$\begin{aligned} \text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)}(1 \otimes e_{ii}) &= E_\omega^\mu(\text{CTr}_{M_\omega^{(\text{id}, \rho)}(E^\mu)}(1 \otimes e_{ii})) = \text{CTr}_{N_\omega^{(\text{id}, \rho)}(E^\mu)}(E_\omega^\mu(1 \otimes e_{ii})) \\ &= \mu_i. \end{aligned}$$

(3) \Rightarrow (4). Since $1 \otimes e_{00}$ and $1 \otimes e_{11}$ have central traces μ_0 and μ_1 , respectively, there exist partial isometries $\{u_j\}_{j=1}^{[\mu_1/\mu_0]+1}$ such that $u_j = (1 \otimes e_{11})u_j(1 \otimes e_{00})$, $u_j^* u_j = (1 \otimes e_{00})$ for $1 \leq j \leq [\mu_1/\mu_0]$ and $\sum_{j=1}^{[\mu_1/\mu_0]+1} u_j u_j^* = 1 \otimes e_{11}$. Let $\{v_j\}_{j=1}^{[\mu_1/\mu_0]+1}$ be as $u_j = v_j \otimes e_{10}$. Take a representing sequence of $v_j = (v_j^\nu)_\nu$ with $(v_j^\nu)^* v_j^\nu = 1$ for $1 \leq j \leq [\mu_1/\mu_0]$ and $\sum_{j=1}^{[\mu_1/\mu_0]+1} v_j^\nu (v_j^\nu)^* = 1$. By Lemma 2.6, v_j^ν satisfies

$$\lim_{\nu \rightarrow \omega} \|\mu_1(\varphi \circ \phi_{\rho_1}) v_j^\nu - \mu_0 v_j^\nu \varphi\| = 0 \quad \text{for all } \varphi \in P_*.$$

This means that $\rho \in \overline{\text{Int}}_{\mu_1/\mu_0}(P)$.

(4) \Rightarrow (5). Take partial isometries $(v_j^\nu)_\nu$, $1 \leq j \leq [\mu_1/\mu_0] + 1$, as Definition 2.4 for $\rho \in \overline{\text{Int}}_{[\mu_1/\mu_0]}(P)$. We set $I := \{1, \dots, [\mu_1/\mu_0] + 1\}$ and $x_j^\nu = v_j^\nu \otimes e_{10}$. Then by Lemma 2.6, the sequence $(x_j^\nu)_\nu$ represents the element x_j in $M_\omega^{(\text{id}, \rho)}(E^\mu)$, we have $\sum_{j \in I} x_j x_j^* = 1$. In particular, $\bigvee_{j \in I} s(x_j x_j^*) = 1$.

(5) \Rightarrow (1). Take a finite family $\{x_j\}_{j \in I} \subset P^\omega$ which satisfies the conditions in (5). The condition (b) implies that $x_j \otimes e_{10} \in M_\omega^{(\text{id}, \rho)}(E^\mu)$. Let $x_j = v_j |x_j|$ be the polar decomposition. Then $v_j \otimes e_{10} \in M_\omega^{(\text{id}, \rho)}(E^\mu)$, and $\bigvee_{j \in I} s(v_j v_j^*) = 1$ from (a). We assume that $z(1 \otimes e_{00}) = 0$ for some $z \in Z(M_\omega^{(\text{id}, \rho)}(E^\mu))$. Then $z(v_j \otimes e_{10}) = (v_j \otimes e_{10})z = 0$, and we have $z(v_j v_j^* \otimes e_{11}) = 0$. Since $\bigvee_{j \in I} s(v_j v_j^*) = 1$, we have $z(1 \otimes e_{11}) = 0$. Hence z must be 0, and equivalently the central support of $1 \otimes e_{00}$ in $M_\omega^{(\text{id}, \rho)}(E^\mu)$ is equal to 1. \square

Now we generalize [29, Proposition 2.3 (ii)] to the case of endomorphisms. Readers are referred to [29, Definition 2.1] for the definition of the approximately inner inclusion of factors.

Theorem 2.9. *Let $\mu_0, \mu_1 > 0$ with $\mu_0 + \mu_1 = 1$. Then the following statements are equivalent:*

- (1) *The inclusion $N^{(\text{id}, \rho)} \overset{E^\mu}{\subset} M^{(\text{id}, \rho)}$ is approximately inner.*
- (2) *The inclusion $\rho(P) \overset{E_\rho}{\subset} P$ is approximately inner and $\rho \in \overline{\text{Int}}_{\mu_1/\mu_0}(P)$.*

If ρ is irreducible, the above statements are also equivalent with the following:

- (3) *The inclusion $\rho(P) \overset{E_\rho}{\subset} P$ is approximately inner and there exists a sequence of partial isometries $(v^\nu)_{\nu=1}^\infty$ in P such that*

$$\lim_{\nu \rightarrow \infty} \|\mu_1(\varphi \circ \phi_\rho)v^\nu - \mu_0 v^\nu \varphi\| = 0 \quad \text{for all } \varphi \in P_*,$$

and v^ν is an isometry (coisometry) when $\mu_0 < \mu_1$ (resp. $\mu_0 \geq \mu_1$).

Proof. (1) \Rightarrow (2). The inclusion $\rho(P) \overset{E_\rho}{\subset} P$ is isomorphic to the reduced inclusion $N^{(\text{id}, \rho)}(1 \otimes e_{11}) \subset (1 \otimes e_{11})M^{(\text{id}, \rho)}(1 \otimes e_{11})$ with a conditional expectation $(E^\mu)_{1 \otimes e_{11}}$.

Hence the inclusion $\rho(P) \overset{E_\rho}{\subset} P$ is approximately inner by [29, Proposition 2.7 (i)].

Next we show that $\rho \in \overline{\text{Int}}_{\mu_1/\mu_0}(P)$. By Proposition 2.8, it suffices to prove that the central support of $1 \otimes e_{00}$ in $M_\omega^{(\text{id}, \rho)}(E^\mu)$ is equal to 1. We make use of a Pimsner-Popa basis for E_ω^μ as follows.

On the corner at $1 \otimes e_{00}$, $m^{00} := 1 \otimes e_{00}$ is a basis because $N_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{00}) = (1 \otimes e_{00})M_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{00})$ holds.

Next we consider the corner at $1 \otimes e_{11}$. By [29, Proposition 2.2], the central sequence inclusion $\rho(P_\omega) \overset{(E_\rho)_\omega}{\subset} P_\omega(E_\rho)$ is a $\text{Ind}(E_\rho)^{-1}$ -Markov inclusion, and so is $N_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{11}) \subset (1 \otimes e_{11})M_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{11})$ with a conditional expectation $(E_\omega^\mu)_{(1 \otimes e_{11})}$. Take an orthonormal basis $(m_j^{11})_{j \in I_{11}}$ for $(E_\omega^\mu)_{(1 \otimes e_{11})}$. Then

$\mu_1^{-1}E_\omega^\mu((m_j^{11})^*m_k^{11}) = \delta_{jk}f_j$ is a projection in $(1 \otimes e_{11})M_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{11})$, and

$$\sum_{j \in I_{11}} m_j^{11}(m_j^{11})^* = \text{Ind}(E_\rho)(1 \otimes e_{11}). \quad (2.1)$$

Now we have an orthonormal family $\{\mu_0^{-1/2}m^{00}\} \cup \{\mu_1^{-1/2}m_j^{11}\}_{j \in I_{11}}$ with respect to the expectation E_ω^μ . By adding other elements $\{m_p\}_{p \in I}$, we can extend the orthonormal family to an orthonormal basis for E_ω .

Then the Markov property of E_ω^μ implies the following equality:

$$\begin{aligned} \text{Ind}(E^\mu) &= \mu_0^{-1}m^{00}(m^{00})^* + \sum_{p \in I} m_p m_p^* + \sum_{j \in I_{11}} \mu_1^{-1}m_j^{11}(m_j^{11})^* \\ &= \mu_0^{-1}(1 \otimes e_{00}) + \sum_{p \in I} m_p m_p^* + \mu_1^{-1} \text{Ind}(E_\rho)(1 \otimes e_{11}) \quad (\text{by (2.1)}). \end{aligned} \quad (2.2)$$

We prove that the central support of $1 \otimes e_{00}$ in $M_\omega^{(\text{id}, \rho)}(E)$ is equal to 1. Assume that $z(1 \otimes e_{00}) = 0$ for a central projection z in $M_\omega^{(\text{id}, \rho)}(E)$. Note that m_p is off-diagonal because m_j^{00} and m_j^{11} are orthonormal bases of the corner of $M_\omega^{(\text{id}, \rho)}(E^\mu)$ reduced by $1 \otimes e_{00}$ and $1 \otimes e_{11}$, respectively. Hence we have $zm_p = 0$. By multiplying z to (2.2), we obtain

$$\text{Ind}(E^\mu)z = \mu_1^{-1} \text{Ind}(E_\rho)z(1 \otimes e_{11}).$$

However the formula of Lemma 2.5 implies that $\text{Ind}(E^\mu) > \mu_1^{-1} \text{Ind}(E_\rho)$. This shows that z must be equal to 0, and the central support of $1 \otimes e_{00}$ is equal to 1.

(2) \Rightarrow (1). We prove that $N_\omega^{(\text{id}, \rho)}(E^\mu) \stackrel{(E^\mu)_\omega}{\subset} M_\omega^{(\text{id}, \rho)}(E^\mu)$ is an $\text{Ind}(E^\mu)^{-1}$ -Markov inclusion. Then $N^{(\text{id}, \rho)} \stackrel{E^\mu}{\subset} M^{(\text{id}, \rho)}$ is approximately inner by [29, Proposition 2.2]. By Proposition 2.8, $1 \otimes e_{00}$ and $1 \otimes e_{11}$ have the scalar central traces μ_0 and μ_1 , respectively. We may assume $\mu_0 < \mu_1$ because the similar proof works in the case of $\mu_0 \geq \mu_1$. This allows us to take a family of partial isometries $\{v_j \otimes e_{10}\}_{j=1}^{m+1}$ in $M_\omega^{(\text{id}, \rho)}(E^\mu)$ such that $v_j^*v_k = 0$ for $j \neq k$, $v_j^*v_j = 1$ for $j \leq m$ and $\sum_{j=1}^{m+1} v_j v_j^* = 1$. We construct a basis for E^μ as follows.

For the e_{00} -entry, we set $m^{00} = \mu_0^{-1/2}(1 \otimes e_{00})$. For the e_{11} -entry, we take a basis $\{m_j^{11}\}_{j \in I_{11}}$ as before.

For the e_{10} -entry, we set $m_j^{10} := (v_j \otimes e_{10})m^{00}$. Then $\{m_j^{10}\}_{j=1}^{m+1}$ is an orthonormal family satisfying

$$\mu_0^{-1}(1 \otimes e_{11}) = \sum_{j=1}^{m+1} m_j^{10}(m_j^{10})^*, \quad (2.3)$$

and

$$(1 \otimes e_{11})M_\omega^{(\text{id}, \rho)}(E^\mu)(1 \otimes e_{00}) = \sum_{j,k=1}^{m+1} m_j^{10} N_\omega^{(\text{id}, \rho)}(E^\mu) m_k^{10}. \quad (2.4)$$

For the e_{01} -entry, we set $m_j^{01} := (v_{i_0}^* \otimes e_{01})m_j^{11}$ for a fixed i_0 . Although $\{m_j^{01}\}_j$ is not an orthonormal family, for any $x \in M_\omega^{(\text{id}, \rho)}(E^\mu)$ we have

$$\begin{aligned} \sum_{j=1}^{m+1} m_j^{01}(E^\mu)_\omega((m_j^{01})^*x) &= \sum_{j=1}^{m+1} (v_{i_0}^* \otimes e_{01})m_j^{11}(E^\mu)_\omega(((v_{i_0}^* \otimes e_{01})m_j^{11})^*x) \\ &= \sum_{j=1}^{m+1} (v_{i_0}^* \otimes e_{01})m_j^{11}(E^\mu)_\omega((m_j^{11})^*(v_{i_0} \otimes e_{10})x(1 \otimes e_{11})) \\ &= (v_{i_0}^* \otimes e_{01})(v_{i_0} \otimes e_{10})x(1 \otimes e_{11}) \\ &= (1 \otimes e_{00})x(1 \otimes e_{11}). \end{aligned}$$

This shows that $(m_j^{01}, (m_j^{01})^*)_j$ is a quasi-basis for e_{01} -entry of $M_\omega^{(\text{id}, \rho)}(E^\mu)$ in the sense of [35]. Moreover we have

$$\sum_{i=1}^{m+1} m_i^{01}(m_i^{01})^* = \mu_1^{-1} \text{Ind}(E_\rho)(1 \otimes e_{00}). \quad (2.5)$$

Then the family $\{(m_i^j, (m_i^j)^*)\}_{i,j}$ is a quasi-basis for $(E^\mu)_\omega$. Using (2.1), (2.3) and (2.5), we have

$$\begin{aligned} \sum_{i,j} m_i^j(m_i^j)^* &= \sum_i m_i^{00}(m_i^{00})^* + \sum_i m_i^{11}(m_i^{11})^* + \sum_i m_i^{10}(m_i^{10})^* + \sum_i m_i^{01}(m_i^{01})^* \\ &= \mu_0^{-1}(1 \otimes e_{00}) + \mu_1^{-1} \text{Ind}(E_\rho)(1 \otimes e_{11}) \\ &\quad + \mu_0^{-1}(1 \otimes e_{11}) + \mu_1^{-1} \text{Ind}(E_\rho)(1 \otimes e_{00}) \\ &= (\mu_0^{-1} + \mu_1^{-1} \text{Ind}(E_\rho))(1 \otimes e_{00} + 1 \otimes e_{11}) \\ &= \text{Ind}(E^\mu) \quad (\text{by Lemma 2.5}). \end{aligned}$$

Hence the inclusion $N_\omega^{(\text{id}, \rho)}(E^\mu) \stackrel{(E^\mu)_\omega}{\subset} M_\omega^{(\text{id}, \rho)}(E^\mu)$ is $\text{Ind}(E^\mu)^{-1}$ -Markov, and the inclusion $N^{(\text{id}, \rho)} \stackrel{E^\mu}{\subset} M^{(\text{id}, \rho)}$ is approximately inner by [29, Proposition 2.2].

(2) \Rightarrow (3). In fact, the irreducibility of ρ is unnecessary. By Proposition 2.8, $e_{00} \otimes 1$ and $e_{11} \otimes 1$ have scalar central traces μ_0 and μ_1 , respectively. If $\mu_0 < \mu_1$, then there exists an isometry $v \in P^\omega$ such that $v \otimes e_{10} \in M_\omega^{(\text{id}, \rho)}(E^\mu)$. Let $(v^\nu)_\nu$ be a representing sequence of v which consists of isometries. By Lemma 2.6, $(v^\nu)_\nu$ satisfies

$$\lim_{\nu \rightarrow \omega} \|\mu_1(\varphi \circ \phi_\rho)v^\nu - \mu_0v^\nu\varphi\| = 0.$$

Then we take a subsequence of $(v^\nu)_\nu$ so that the above equality holds as $\nu \rightarrow \infty$.

If $\mu_0 \geq \mu_1$, then the similar argument still works, and we can find a desired sequence which consists of coisometries.

(3) \Rightarrow (2). We prove that the central support of $1 \otimes e_{00}$ is equal to 1. Take a sequence of partial isometries $(v^\nu)_\nu$ as in (3). Put $v := \pi_\omega((v^\nu)_\nu) \in P^\omega$. Then the element $v \otimes e_{10}$ is contained in $M_\omega^{(\text{id}, \rho)}(E^\mu)$ by Lemma 2.6.

If $\mu_0 \geq \mu_1$, we have $1 \otimes e_{11} = (v \otimes e_{10})(v \otimes e_{10})^*$ which is equivalent to $v^*v \otimes e_{00}$. Hence the central support of $1 \otimes e_{00}$ is equal to 1 in this case.

If $\mu_0 < \mu_1$, we have $1 \otimes e_{00} = (v \otimes e_{10})^*(v \otimes e_{10})$. Suppose that a non-zero projection $z \in Z(M_\omega^{(\text{id}, \rho)}(E^\mu))$ satisfies $z(1 \otimes e_{00}) = 0$. Then z is of the form $z = z_1 \otimes e_{11}$, where z_1 is a projection in $Z(P_\omega(E_\rho))$. Since z is central, we have $z(v \otimes e_{10}) = (v \otimes e_{10})z = (v \otimes e_{10})(1 \otimes e_{00})z = 0$. This means $z_1 v v^* = 0$. Note that $v v^* \in P_\omega(E_\rho)$, and $\tau^\omega(v v^*) \in \rho(P)' \cap P = \mathbb{C}$ because ρ is irreducible. Now for z_1 and v , we apply the Fast Reindexation Lemma [28, Lemma 5.3]. Then there exists a map $\Psi: W^*(v) \rightarrow P^\omega$ such that $\tau^\omega(z_1 \Psi(v v^*)) = \tau^\omega(z_1) \tau^\omega(v v^*)$, which is not equal to 0. In particular, $z_1 \Psi(v) \neq 0$. By the construction of the fast reindexation map Ψ in the proof of [28, Lemma 5.3], we may assume that $\Psi(v)$ is given by mapping $(v^\nu)_\nu$ to some subsequence $(v^{\nu(k)})_k$. Hence $\Psi(v) \otimes e_{10}$ is also an element of $M_\omega^{(\text{id}, \rho)}(E^\mu)$. However, $z_1 \Psi(v) \otimes e_{10} = z(\Psi(v) \otimes e_{10}) = (\Psi(v) \otimes e_{10})z = 0$, and this is a contradiction.

Therefore in the both cases, the central support of $1 \otimes e_{00}$ is equal to 1. Then by Proposition 2.8, $\rho \in \overline{\text{Int}}_{\mu_1/\mu_0}(P)$. \square

2.5. Basic properties of approximately inner endomorphisms

Let M be a factor and $\rho \in \overline{\text{Int}}_r(M)$ with $r > 0$. Then for any $u \in U(M)$, $\text{Ad } u \circ \rho$ is also in $\overline{\text{Int}}_r(M)$. Hence approximate innerness is a property for sectors. The sector space $\text{Sect}(M)$ has the basic operations, i.e., composition, decomposition, direct sum and conjugation. We study how approximate innerness behaves for these operations. For sector theory, readers are referred to [15, 21, 22].

Lemma 2.10 (Decomposition rule). *Let $\rho \in \text{End}(M)_0$ and $[\rho] = \oplus_{i \in I} m_i [\rho_i]$ be the irreducible decomposition where m_i is the multiplicity of $[\rho_i]$ in $[\rho]$. Then $\rho \in \overline{\text{Int}}_r(M)$ if and only if $\rho_i \in \overline{\text{Int}}_{rd(\rho_i)/d(\rho)}(M)$ for all $i \in I$.*

Proof. Suppose that $\rho \in \overline{\text{Int}}_r(M)$. Take $\{v_j^\nu\}_{j=1}^{[r]+1}$, $\nu \in \mathbb{N}$, as in Definition 2.4. Let $\{w_k^i\}_{k=1}^{m_i}$ be an orthonormal basis of the Hilbert space (ρ_i, ρ) . Then we have

$$d(\rho)\phi_\rho(x) = \sum_{i \in I} \sum_{k=1}^{m_i} d(\rho_i)\phi_{\rho_i}((w_k^i)^* x w_k^i).$$

For the proof of this equality, readers are referred to [23, Lemma A.2]. Then for all $\varphi \in M_*$ and $1 \leq k \leq m_i$, we have

$$\lim_{\nu \rightarrow \infty} \left\| \frac{1}{r} ((w_k^i)^* v_j^\nu) \cdot \varphi - \frac{d(\rho_i)}{d(\rho)} \varphi \circ \phi_{\rho_i} \cdot ((w_k^i)^* v_j^\nu) \right\| = 0.$$

It is equivalent to $\pi_\omega(((w_k^i)^* v_j^\nu)_\nu) \otimes e_{10} \in M_\omega^{(\text{id}, \rho_i)}(E^{(\mu_0^i, \mu_1^i)})$, where $\mu_0^i + \mu_1^i = 1$ and $\mu_0^i/\mu_1^i = d(\rho)/(rd(\rho_i))$. Setting $x_{k,j}^{i,\nu} := (w_k^i)^* v_j^\nu$, we have

$$\sum_{j=1}^{[r]+1} \sum_{k=1}^{m_i} x_{k,j}^{i,\nu} (x_{k,j}^{i,\nu})^* = \sum_{j=1}^{[r]+1} \sum_{k=1}^{m_i} (w_k^i)^* v_j^\nu (v_j^\nu)^* w_k^i = \sum_{k=1}^{m_i} (w_k^i)^* w_k^i = \dim(\rho_i, \rho).$$

By Proposition 2.8, we see that $\rho_i \in \overline{\text{Int}}_{rd(\rho_i)/d(\rho)}(M)$.

Conversely we suppose that $\rho_i \in \overline{\text{Int}}_{rd(\rho_i)/d(\rho)}(M)$ for all $i \in I$. For each $i \in I$, we take sequences of partial isometries $\{v_j^{i\nu}\}_{j=1}^{[rd(\rho_i)/d(\rho)]+1}$, $\nu \in \mathbb{N}$, as in Definition 2.4. Then for all i, j and $\varphi \in M_*$, it satisfies

$$\lim_{\nu \rightarrow \infty} \left\| \frac{1}{r} v_j^{i\nu} \varphi - \frac{d(\rho_j)}{d(\rho)} (\varphi \circ \phi_{\rho_i}) \cdot v_j^{i\nu} \right\| = 0.$$

Then for all $1 \leq k \leq m_i$,

$$\lim_{\nu \rightarrow \infty} \left\| r^{-1} w_k^i v_j^{i\nu} \varphi - (\varphi \circ \phi_\rho) \cdot (w_k^i v_j^{i\nu}) \right\| = 0.$$

Hence $\pi_\omega((w_k v_j^{i\nu})_\nu) \otimes e_{10} \in M_\omega^{(\text{id}, \rho)}(E^{(\mu_0, \mu_1)})$, where $\mu_0 + \mu_1 = 1$ and $\mu_0/\mu_1 = 1/r$.

Since

$$\sum_{i \in I} \sum_{k=1}^{m_i} \sum_{j=1}^{[rd(\rho_i)/d(\rho)]+1} w_k^i v_j^{i\nu} (w_k^i v_j^{i\nu})^* = 1,$$

$\rho \in \overline{\text{Int}}_r(M)$ by Proposition 2.8. \square

Corollary 2.11. *Let $\rho, \sigma \in \text{End}(M)_0$. Suppose that $\sigma \prec_r \rho$ and $\rho \in \overline{\text{Int}}_r(M)$. Then $\sigma \in \overline{\text{Int}}_{rd(\sigma)/d(\rho)}(M)$.*

Proof. Let $[\sigma] = \oplus_i [\sigma_i]$ be the irreducible decomposition. By applying the previous lemma to $\sigma_i \prec_r \rho$, we have $\sigma_i \in \overline{\text{Int}}_{rd(\sigma_i)/d(\rho)}(M)$. Note that $rd(\sigma_i)/d(\rho) = (rd(\sigma)/d(\rho))d(\sigma_i)/d(\sigma)$. Using again the previous lemma, we see that $\sigma \in \overline{\text{Int}}_{rd(\sigma)/d(\rho)}(M)$. \square

On composition of endomorphisms, the following result holds.

Lemma 2.12 (Composition rule). *Let $\rho_i \in \overline{\text{Int}}_{r_i}(N)$ for $i = 1, 2$. Then $\rho_1 \circ \rho_2 \in \overline{\text{Int}}_{r_1 r_2}(N)$.*

Proof. Take sequences of partial isometries $\{v_j^{1\nu}\}_{j=1}^{[r_1]+1}$ and $\{v_j^{2\nu}\}_{j=1}^{[r_2]+1}$ satisfying the conditions in Definition 2.4 for ρ_1 and ρ_2 , respectively. Then for all $\varphi \in M_*$, $i = 1, 2$ and $1 \leq j \leq [r_i] + 1$,

$$\lim_{\nu \rightarrow \infty} \left\| r_i^{-1} v_j^{i\nu} \varphi - (\varphi \circ \phi_{\rho_i}) \cdot v_j^{i\nu} \right\| = 0.$$

It is easy to see that

$$\lim_{\nu \rightarrow \infty} \left\| (r_1 r_2)^{-1} v_j^{1\nu} v_k^{2\nu} \varphi - (\varphi \circ \phi_{\rho_2} \circ \phi_{\rho_1}) \cdot v_j^{1\nu} v_k^{2\nu} \right\| = 0$$

for all $1 \leq j \leq [r_1] + 1$ and $1 \leq k \leq [r_2] + 1$. Hence $\pi_\omega((v_j^{1\nu} v_k^{2\nu})_\nu) \otimes e_{10} \in M_\omega^{(\text{id}, \rho_1 \rho_2)}(E^{(\mu_0, \mu_1)})$, where $\mu_0 + \mu_1 = 1$ and $\mu_0/\mu_1 = 1/(r_1 r_2)$. Since

$$1 = \sum_{j=1}^{[r_1]+1} \sum_{k=1}^{[r_2]+1} v_j^{1\nu} v_k^{2\nu} (v_j^{1\nu} v_k^{2\nu})^*,$$

$\rho_1 \circ \rho_2 \in \overline{\text{Int}}_{r_1 r_2}(M)$ by Proposition 2.8. \square

On conjugation, we have a result only for hyperfinite factors (Corollary 3.18).

2.6. Descriptions of $\overline{\text{Int}}_r(N)$ for hyperfinite semifinite factors

Lemma 2.13. *Let N be a type I factor, then $\overline{\text{Int}}_r(N) = \emptyset$ for all $r \notin \mathbb{N}$ and $\overline{\text{Int}}_r(N) = \text{Int}_r(N)$ for all $r \in \mathbb{N}$.*

Proof. Suppose $\rho \in \text{Int}_d(N) \cap \overline{\text{Int}}_r(N)$ for some $r > 0$, where $d = d(\rho)$. We will show $r = d$. Take sequences of partial isometries $\{v_i^\nu\}_{i=1}^{[r]+1} \subset N$, $\nu \in \mathbb{N}$ such that $(v_i^\nu)^* v_i^\nu = 1$ for $1 \leq i \leq [r]$, $\sum_{i=1}^{[r]+1} v_i^\nu (v_i^\nu)^* = 1$ for all $\nu \in \mathbb{N}$ and

$$\lim_{\nu \rightarrow \infty} \|r^{-1} v_i^\nu \varphi - (\varphi \circ \phi_\rho) v_i^\nu\| = 0 \quad (2.6)$$

for all $1 \leq i \leq [r] + 1$, $\varphi \in N_*$. Take a Hilbert space $\mathcal{H} \subset N$ implementing ρ . Let $\{w_j\}_{j=1}^d$ be an orthonormal basis of \mathcal{H} . Then $\phi_\rho(x) = d^{-1} \sum_{j=1}^d w_j^* x w_j$ for $x \in N$. Hence (2.6) is equivalent with

$$\lim_{\nu \rightarrow \infty} \|r^{-1} (w_j^* v_i^\nu) \cdot \varphi - d^{-1} \varphi \cdot (w_j^* v_i^\nu)\| = 0$$

for all $1 \leq i \leq [r] + 1$, $1 \leq j \leq d$, $\varphi \in N_*$.

Let $a \in N$ be a trace class operator. We apply the above limit equality to $\varphi = \tau_a$. Then the trace norm of $r^{-1} w_j^* v_i^\nu a - d^{-1} a w_j^* v_i^\nu$ converges to 0. Since the trace norm dominates the uniform norm, we have

$$\lim_{\nu \rightarrow \infty} \|r^{-1} w_j^* v_i^\nu a - d^{-1} a w_j^* v_i^\nu\| = 0.$$

Let a be a finite projection, and we have

$$\lim_{\nu \rightarrow \infty} \|(r^{-1} - d^{-1}) a w_j^* v_i^\nu a\| = 0, \quad \lim_{\nu \rightarrow \infty} \|(1 - a) w_j^* v_i^\nu a\| = 0, \quad \lim_{\nu \rightarrow \infty} \|a w_j^* v_i^\nu (1 - a)\| = 0.$$

If $r \neq d$, then the above equalities imply that $\|w_j^* v_i^\nu a\| \rightarrow 0$ and $\|a w_j^* v_i^\nu\| \rightarrow 0$ as $\nu \rightarrow \infty$ for a finite projection $a \in N$. Thus $w_j^* v_i^\nu \rightarrow 0$ strongly* as $\nu \rightarrow \infty$. Since $v_i^\nu = \sum_{j=1}^d w_j (w_j^* v_i^\nu)$, $v_i^\nu \rightarrow 0$ strongly* as $\nu \rightarrow \infty$. This is a contradiction with $1 = \sum_{i=1}^{[r]+1} v_i^\nu (v_i^\nu)^*$. Hence $r = d$, and $\text{Int}_d(N) \cap \overline{\text{Int}}_r(N) \neq \emptyset$ yields $r = d$.

Since any endomorphism on N is inner, the statement of this lemma holds. \square

Lemma 2.14. *If N is a type II_1 factor with the tracial state τ , then $\overline{\text{Int}}_r(N) = \emptyset$ for all $r \neq 1$. Moreover if N is a hyperfinite factor, then $\overline{\text{Int}}_1(N) = \{\rho \in \text{End}(N)_0 \mid \tau \circ \phi_\rho = \tau\}$.*

Proof. If $\overline{\text{Int}}_r(N) \neq \emptyset$, we can take $\rho \in \overline{\text{Int}}_r(N)$. By definition, there exist sequences of partial isometries $\{v_i^\nu\}_{i=1}^{[r]+1} \subset N$, $\nu \in \mathbb{N}$, with the conditions in Definition 2.4. At least v_1^ν is an isometry (or coisometry) when $r \geq 1$ (resp. $0 < r < 1$), but we note that any isometry (or coisometry) is a unitary because N is finite. Hence v_1^ν is unitary. Then we have

$$\lim_{n \rightarrow \infty} \|r^{-1} \varphi \circ \text{Ad}(v_1^\nu)^* - \varphi \circ \phi_\rho\| = 0 \quad \text{for all } \varphi \in N_*.$$

In particular, $r^{-1} \varphi(1) - \varphi(1) = (r^{-1} \varphi \circ \text{Ad}(v_1^\nu)^* - \varphi \circ \phi_\rho)(1)$ is equal to 0. Hence r must be equal to 1. The latter assertion follows from [24, Lemma 3.9]. \square

Lemma 2.15. *Let N be the hyperfinite type II_∞ factor of with the trace τ . Let $\rho \in \text{End}(N)_0$ and $\text{mod}(\rho)$ be the module of ρ , i.e., $\tau \circ \phi_\rho = d(\rho)^{-1} \text{mod}(\rho)^{-1} \tau$. Let $\lambda > 0$. Then $\rho \in \overline{\text{Int}}_\lambda(N)$ if and only if $\lambda = d(\rho) \text{mod}(\rho)$.*

Proof. We will show $\rho \in \overline{\text{Int}}_\lambda(N)$ for $\lambda = d(\rho) \text{mod}(\rho)$. Set $\mu_0 = (1 + \lambda)^{-1}$, $\mu_1 = \lambda(1 + \lambda)^{-1}$. Consider the locally trivial subfactor $N^{(\text{id}, \rho)} \subset M^{(\text{id}, \rho)}$ with the conditional expectation $E := E^{(\mu_0, \mu_1)}$. Then E preserves the trace $\tau \otimes \text{Tr}$ on $M^{(\text{id}, \rho)}$, and the locally trivial inclusion $N^{(\text{id}, \rho)} \xrightarrow{E^\mu} M^{(\text{id}, \rho)}$ is approximately inner by [29, Theorem 2.9 (i)]. Then Theorem 2.9 implies that $\rho \in \overline{\text{Int}}_\lambda(N)$.

Conversely we assume $\rho \in \overline{\text{Int}}_\lambda(N)$ for some $\lambda > 0$. We set $\mu_0 := (1 + \lambda)^{-1}$ and $\mu_1 := \lambda(1 + \lambda)^{-1}$. Then the expectation E_ρ preserves τ , and $\rho(N) \xrightarrow{E_\rho} N$ is approximately inner [29, Theorem 2.9 (i)]. Hence the locally trivial subfactor $N^{(\text{id}, \rho)} \xrightarrow{E^\mu} M^{(\text{id}, \rho)}$ is approximately inner by Theorem 2.9. Again by [29, Theorem 2.9 (i)], E^μ preserves the trace $\tau \otimes \text{Tr}$, that is, $\lambda = d(\rho) \text{mod}(\rho)$. \square

We will use the following generalization of the previous lemma to non-factorial case. The definition of approximate innerness is naturally extended to this case as Definition 2.4, but we have to fix a left inverse of an endomorphism.

Lemma 2.16. *Let N be a hyperfinite type II_∞ von Neumann algebra with a faithful normal semifinite trace τ . Let $\rho \in \text{End}(N)$ with a left inverse ϕ_ρ . If $\rho|_{Z(N)} = \text{id}$ and $\tau \circ \phi_\rho = \lambda^{-1} \tau$ for some $\lambda > 0$, then ρ is approximately inner of rank λ with respect to ϕ_ρ .*

Proof. By assumption on the hyperfiniteness, we can regard $N = Z(N) \otimes R_{0,1}$.

First we assume that $\lambda = 1$. We have $\tau \circ \phi_\rho = \tau$, and $\tau \circ \rho = \tau$. Since $\rho = \text{id}$ on $Z(N)$, $\rho(p)$ is equivalent to p for any projection $p \in N$.

Take a system of matrix units $\{e_{i,j}\}_{i,j=1}^\infty$ in $R_{0,1}$ such that $e_{i,i}$ are finite projections for all i . We take a partial isometry $w \in N$ such that $w^*w = 1 \otimes e_{1,1}$ and $ww^* = \rho(1 \otimes e_{11})$. Then we set a unitary $v := \sum_{i=1}^\infty \rho(1 \otimes e_{i1})w(1 \otimes e_{i1})$. It is easy to see that $v(1 \otimes e_{ij}) = \rho(1 \otimes e_{ij})v$ for all i, j . Hence $\sigma := \text{Ad } v^* \circ \rho$ fixes $Z(N)$ and the type I subfactor B generated by $\{1 \otimes e_{i,j}\}_{i,j=1}^\infty$. Considering the left inverse $\phi_\sigma := \phi_\rho \circ \text{Ad } v$ of σ , we may assume that ρ fixes B . We note that ϕ_ρ also fixes them. Indeed if $\rho(x) = x$ for $x \in N$, then $\phi_\rho(x) = \phi_\rho(\rho(x)) = x$.

Now consider the reduced endomorphism $\rho^{1 \otimes e_{11}}$ on the hyperfinite type II_1 von Neumann algebra $(1 \otimes e_{11})N(1 \otimes e_{11}) = Z(N) \otimes e_{11}R_{0,1}e_{11}$. Using the natural isomorphism from $R_{0,1}$ onto $e_{11}R_{0,1}e_{11} \otimes B$, we see that ρ and ϕ_ρ are of the form $\rho^{1 \otimes e_{11}} \otimes \text{id}_B$ and $\phi_\rho^{1 \otimes e_{11}} \otimes \text{id}_B$ on $Z(N) \otimes e_{11}R_{0,1}e_{11} \otimes B$, respectively. Then the same proof of [24, Lemma 3.9] works after a slight modification on a treatment of the center. Hence $\rho^{1 \otimes e_{11}}$ is approximately inner of rank 1, and so is ρ .

Second we consider a general case. Take $\theta \in \text{Aut}(R_{0,1})$ with module λ . By the previous lemma, $\theta \in \text{Aut}(R_{0,1})$ is approximately inner of rank λ . The trace τ is given by $\tau_0 \otimes \tau_1$ where τ_0 and τ_1 are the traces on $Z(N)$ and $R_{0,1}$, respectively. Then the automorphism $\text{id} \otimes \theta$ satisfies $\tau \circ (\text{id} \otimes \theta) = \lambda \tau$. We simply write $\text{id} \otimes \theta$ as θ below. It is easy to see that $\theta \in \text{Aut}(N)$ is also approximately inner of

rank λ with respect to the left inverse θ^{-1} . Obviously θ is trivial on $Z(N)$. Set $\rho_0 := \theta^{-1}\rho$ and $\phi_{\rho_0} = \phi_\rho \circ \theta$. Then we have $\rho_0|_{Z(N)} = \text{id}$ and $\tau \circ \phi_{\rho_0} = \tau$, and the first part of the proof implies that ρ_0 is approximately inner of rank 1.

Take sequences of partial isometries $\{v_j^\nu\}_{j=1}^{[\lambda]+1}$, $\{u^\nu\}_\nu$, $\nu \in \mathbb{N}$, in N such that

- (1) $(v_j^\nu)^* v_j^\nu = 1$ for $1 \leq j \leq [\lambda]$, $\sum_{j=1}^{[\lambda]+1} v_j^\nu (v_j^\nu)^* = 1$.
- (2) $(u^\nu)^* u^\nu = 1 = u^\nu (u^\nu)^*$.
- (3) For all $\varphi \in N_*$ and $1 \leq j \leq [\lambda] + 1$,

$$\lim_{\nu \rightarrow \infty} \|\lambda^{-1} v_j^\nu \varphi - (\varphi \circ \theta^{-1}) \cdot v_j^\nu\| = 0, \quad \lim_{\nu \rightarrow \infty} \|u^\nu \varphi - (\varphi \circ \phi_{\rho_0}) \cdot u^\nu\| = 0.$$

Then it is easy to see that

$$\lim_{\nu \rightarrow \infty} \|\lambda^{-1} v_j^\nu u^\nu \varphi - (\varphi \circ \phi_\rho) \cdot v_j^\nu u^\nu\| = 0.$$

Since $v_j^\nu u^\nu$ is an isometry for $1 \leq j \leq [\lambda]$ and $\sum_{j=1}^{[\lambda]+1} v_j^\nu u^\nu (v_j^\nu u^\nu)^* = 1$, ρ is approximately inner of rank λ by definition. \square

3. CANONICAL EXTENSIONS AND APPROXIMATELY INNER ENDOMORPHISMS

In this section, we discuss a generalization of the result proved by Kawahigashi, Sutherland and Takesaki [19], that was first announced by Connes without a proof [5]. Their result says that for any hyperfinite factor M , an automorphism on M is approximately inner if and only if it has trivial Connes-Takesaki module, that is,

$$\overline{\text{Int}}(M) = \text{Ker}(\text{mod}).$$

3.1. Canonical extension

We recall canonical extensions of endomorphisms introduced by Izumi [17].

Let M be a factor and \widetilde{M} the canonical core extension of M [8, Definition 2.5], which is the von Neumann algebra generated by M and one-parameter unitary groups $\{\lambda^\varphi(t)\}_{t \in \mathbb{R}}$, $\varphi \in W(M)$, satisfying the relations

$$\sigma_t^\varphi(x) = \lambda^\varphi(t) x \lambda^\varphi(t)^*, \quad \lambda^\psi(t) = [D\psi : D\varphi]_t \lambda^\varphi(t)$$

for all $x \in M$, $t \in \mathbb{R}$ and $\varphi, \psi \in W(M)$.

Let us represent M on a Hilbert space H . The crossed product $M \rtimes_{\sigma^\varphi} \mathbb{R}$ for the modular automorphism group σ^φ is the von Neumann algebra generated by $\pi_{\sigma^\varphi}(M)$ and $\lambda(\mathbb{R})$ in $B(H \otimes L^2(\mathbb{R}))$ such that

$$(\pi_{\sigma^\varphi}(x)\xi)(s) = \sigma_{-s}^\varphi(x)\xi(s), \quad (\lambda(t)\xi)(s) = \xi(-t+s)$$

for all $x \in M$, $\xi \in H \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}, H)$ and $s, t \in \mathbb{R}$. By [8, Theorem 2.4], we have the isomorphism $\Pi_\varphi: \widetilde{M} \rightarrow M \rtimes_{\sigma^\varphi} \mathbb{R}$ satisfying

$$\Pi_\varphi(x) = \pi_{\sigma^\varphi}(x), \quad \Pi_\varphi(\lambda^\varphi(t)) = \lambda(t) \quad \text{for all } x \in M, t \in \mathbb{R}.$$

Let θ be the \mathbb{R} -action on \widetilde{M} satisfying

$$\theta_s(x) = x, \quad \theta_s(\lambda^\varphi(t)) = e^{-ist} \lambda^\varphi(t) \quad \text{for all } x \in M, s, t \in \mathbb{R}.$$

Then $\Pi_\varphi \circ \theta_s = \widehat{\sigma^\varphi}_s \circ \Pi_\varphi$ for all $s \in \mathbb{R}$. The action θ is also called the dual action.

Let ρ be an endomorphism on M with finite index. Then the *canonical extension* $\tilde{\rho}$ of ρ is the endomorphism on \widetilde{M} defined by

$$\tilde{\rho}(x) = \rho(x), \quad \tilde{\rho}(\lambda^\varphi(t)) = d(\rho)^{it} [D\varphi \circ \phi_\rho : D\varphi]_t \lambda^\varphi(t)$$

for all $x \in M$, $t \in \mathbb{R}$ and $\varphi \in W(M)$. Note that $\tilde{\rho}$ commutes the dual action θ .

3.2. Normalized canonical extension

Let $\sim : \text{End}(M)_0 \rightarrow \text{End}(\widetilde{M})$ be the canonical extension. It is known that the canonical extension is continuous on $\text{Aut}(M)$ with respect to the u -topology. However in general, it is not continuous on $\text{End}(M)_0$ because the statistical dimension map $d : \text{End}(M)_0 \rightarrow [1, \infty)$ is not continuous with respect to our topology (recall Definition 2.1). In §3.4, we will discuss a relation between approximate innerness and Connes-Takesaki modules. Then we need the continuity for that purpose. Hence we introduce a modified canonical extension map as follows.

Definition 3.1. Let M be a factor and $\rho \in \text{End}(M)_0$. We define the *normalized canonical extension map* $\check{\rho} : \text{End}(M)_0 \rightarrow \text{End}(\widetilde{M})$ by

$$\check{\rho}(x) = \rho(x), \quad \check{\rho}(\lambda^\varphi(t)) = [D\varphi \circ \phi_\rho : D\varphi]_t \lambda^\varphi(t)$$

for all $x \in M$, $t \in \mathbb{R}$ and $\varphi \in W(M)$.

Indeed, we have

$$\check{\rho} = \theta_{\log(d(\rho))} \circ \tilde{\rho} = \tilde{\rho} \circ \theta_{\log(d(\rho))},$$

which shows the existence of $\check{\rho}$.

Next we want to discuss a convergence in $\text{End}(\widetilde{M})$ by using particular left inverses. Note that \widetilde{M} may not be a factor. Even in non-factorial case, minimal expectations can be also defined as in [9], and it will be possible to give a topology. However, in order to avoid using a disintegration of factors and left inverses, we do not take such a way. For our purpose, the following notion presented in [24, Definition 3.1] is sufficient.

Definition 3.2. Let M be a von Neumann algebra and ρ^ν , $\nu \in \mathbb{N}$, ρ endomorphisms on M . Let ϕ^ν and ϕ be left inverses of ρ^ν and ρ , respectively. We say that the sequence of the pairs $\{(\rho^\nu, \phi^\nu)\}_\nu$ converges to (ρ, ϕ) if

$$\lim_{\nu \rightarrow \infty} \|\varphi \circ \phi^\nu - \varphi \circ \phi\| = 0 \quad \text{for all } \varphi \in M_*.$$

Note that the above convergence implies pointwise strong* convergence, that is, if (ρ^ν, ϕ^ν) converges to (ρ, ϕ) , then $\rho^\nu(x) \rightarrow \rho(x)$ strongly* as $\nu \rightarrow \infty$ for any $x \in M$ [24, Lemma 3.8]. When M is a factor, ρ^ν converges to ρ in $\text{End}_0(M)$ in the topology defined in Definition 2.1 if and only if $(\rho^\nu, \phi_{\rho^\nu})$ converges to (ρ, ϕ_ρ) . We study the relationship between the convergence of endomorphisms and that of implementing isometries [10].

Let M be a factor as before. We represent M on the standard Hilbert space $L^2(M)$. The positive cone is denoted by $L^2(M)_+$. In what follows, we use the

following useful equalities for $\rho \in \text{End}(M)_0$:

$$\sigma_t^{\psi \circ \phi_\rho} \circ \rho = \rho \circ \sigma_t^\psi, \quad [D\psi \circ \phi_\rho : D\chi \circ \phi_\rho]_t = \rho([D\psi : D\chi]_t) \quad (3.1)$$

for all $\psi, \chi \in W(M)$ and $t \in \mathbb{R}$ [17, p.5–7]. Since $\psi \circ \phi_\rho \circ E_\rho = \psi \circ \phi_\rho$, E_ρ and $\sigma_t^{\psi \circ \phi_\rho}$ commute [33, p.317]. Hence we also have

$$\sigma_t^{\psi \circ \phi_\rho} \circ E_\rho = E_\rho \circ \sigma_t^{\psi \circ \phi_\rho}, \quad \sigma_t^\psi \circ \phi_\rho = \phi_\rho \circ \sigma_t^{\psi \circ \phi_\rho}. \quad (3.2)$$

where the latter equality follows from the former one and (3.1).

Now let us fix a faithful state $\psi \in M_*$. We take a unit vector $\xi_\psi \in L^2(M)_+$ such that $\psi(x) = (x\xi_\psi, \xi_\psi)$ for $x \in M$. For each $\rho \in \text{End}(M)_0$, we take a unit vector $\xi_{\psi \circ \phi_\rho} \in L^2(M)_+$ such that $\psi(\phi_\rho(x)) = (x\xi_{\psi \circ \phi_\rho}, \xi_{\psi \circ \phi_\rho})$ for $x \in M$. Following [10, Appendix A], we define the standard implementation V_ρ for ρ by

$$V_\rho(x\xi_\psi) = \rho(x)\xi_{\psi \circ \phi_\rho} \quad \text{for all } x \in M.$$

Then the isometry V_ρ satisfies the following [10, Proposition A.2]:

$$V_\rho x = \rho(x)V_\rho, \quad \phi_\rho(x) = V_\rho^* x V_\rho \quad \text{for all } x \in M.$$

Let Δ_ψ and $\Delta_{\psi \circ \phi_\rho}$ be the modular operators of ψ and $\psi \circ \phi_\rho$, respectively. Then,

$$V_\rho \Delta_\psi^{it} = \Delta_{\psi \circ \phi_\rho}^{it} V_\rho \quad \text{for all } t \in \mathbb{R}. \quad (3.3)$$

Indeed, using a formula $\sigma_t^{\psi \circ \phi_\rho} \circ \rho = \rho \circ \sigma_t^\psi$, we have

$$\begin{aligned} V_\rho \Delta_\psi^{it}(x\xi_\psi) &= V_\rho(\sigma_t^\psi(x)\xi_\psi) = \rho(\sigma_t^\psi(x))\xi_{\psi \circ \phi_\rho} \\ &= \sigma_t^{\psi \circ \phi_\rho}(\rho(x))\xi_{\psi \circ \phi_\rho} = \Delta_{\psi \circ \phi_\rho}^{it}\rho(x)\xi_{\psi \circ \phi_\rho} \\ &= \Delta_{\psi \circ \phi_\rho}^{it} V_\rho(x\xi_\psi). \end{aligned}$$

Lemma 3.3. *Let $(\rho^\nu)_{\nu \in \mathbb{N}}$ and ρ be endomorphisms on a factor M with finite index. Then the following statements are equivalent.*

- (1) $(\rho^\nu)_\nu$ converges to ρ .
- (2) $(\rho^\nu(x))_\nu$ converges to $\rho(x)$ strongly* for all $x \in M$ and $\lim_{\nu \rightarrow \infty} \xi_{\psi \circ \phi_{\rho^\nu}} = \xi_{\psi \circ \phi_\rho}$.
- (3) $(V_{\rho^\nu})_\nu$ converges to V_ρ strongly.

Proof. (1) \Rightarrow (2). The strong* convergence of $\rho^\nu(x)$ follows from [24, Lemma 3.8]. Since M acts on $L^2(M)$ standardly, the convergence $\psi \circ \phi_{\rho^\nu} \rightarrow \psi \circ \phi_\rho$ implies the convergence $\xi_{\psi \circ \phi_{\rho^\nu}} \rightarrow \xi_{\psi \circ \phi_\rho}$ [11, Lemma 2.10].

(2) \Rightarrow (3). By (2), we see that for all $x \in M$,

$$\lim_{\nu \rightarrow \infty} V_{\rho^\nu}(x\xi_\psi) = \lim_{\nu \rightarrow \infty} \rho^\nu(x)\xi_{\psi \circ \phi_{\rho^\nu}} = \rho(x)\xi_{\psi \circ \phi_\rho} = V_\rho(x\xi_\psi).$$

The norm-boundedness of V_{ρ^ν} implies the strong convergence.

(3) \Rightarrow (1). For vectors $\xi, \eta \in L^2(M)$, we denote by $\omega_{\xi, \eta} \in M_*$ the functional $\omega_{\xi, \eta}(x) = (x\xi, \eta)$ for $x \in M$. Since V_{ρ^ν} implements ϕ_{ρ^ν} , we have $\omega_{\xi, \eta} \circ \phi_{\rho^\nu} = \omega_{V_{\rho^\nu}\xi, V_{\rho^\nu}\eta}$. By elementary calculation, we have

$$\|\omega_{\xi, \eta} \circ \phi_{\rho^\nu} - \omega_{\xi, \eta} \circ \phi_\rho\| \leq \|\eta\| \|V_{\rho^\nu}\xi - V_\rho\xi\| + \|\xi\| \|V_{\rho^\nu}\eta - V_\rho\eta\|.$$

Hence we have the norm convergence $\omega_{\xi,\eta} \circ \phi_{\rho^\nu} \rightarrow \omega_{\xi,\eta} \circ \phi_\rho$ as $\nu \rightarrow \infty$. Since any normal functional on M is of the form $\omega_{\xi,\eta}$, $\xi, \eta \in L^2(M)$, we have done. \square

Lemma 3.4. *Let M be an infinite factor, $\rho_1, \rho_2 \in \text{End}(M)_0$ and $v_1, v_2 \in M$ isometries with $v_1 v_1^* + v_2 v_2^* = 1$. We define $\rho \in \text{End}(M)$ by $\rho(x) := v_1 \rho_1(x) v_1^* + v_2 \rho_2(x) v_2^*$. Then for any weight φ on M , we have*

$$d(\rho)^{it} [D\varphi \circ \phi_\rho : D\varphi]_t = \sum_{k=1}^2 d(\rho_k)^{it} v_k [D\varphi \circ \phi_{\rho_k} : D\varphi]_t \sigma_t^\varphi(v_k^*).$$

Proof. It is shown by using $d(\rho)\phi_\rho(x) = \sum_{k=1}^2 d(\rho_k)\phi_{\rho_k}(v_k^* x v_k)$ and the relative modular condition [34, Theorem VIII.3.3]. \square

Now we construct a left inverse of a canonical extension.

Lemma 3.5. *Let $\rho \in \text{End}(M)_0$ and $\tilde{\rho}$ be the canonical extension of ρ . Then there exists a left inverse $\phi_{\tilde{\rho}}$ on \widetilde{M} such that*

$$\phi_{\tilde{\rho}}(x\lambda^\varphi(t)) = d(\rho)^{-it} \phi_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t) \lambda^\varphi(t)$$

for all $x \in M$, $t \in \mathbb{R}$ and $\varphi \in W(M)$.

Proof. When M is finite, we consider $P := B(\ell^2) \otimes M$ and $\sigma := \text{id} \otimes \rho$. Then $\tilde{P} = B(\ell^2) \otimes \widetilde{M}$ and $\tilde{\sigma} = \text{id} \otimes \tilde{\rho}$. If the statement holds for infinite case, there exists a left inverse $\phi_{\tilde{\sigma}}$ on \tilde{P} with the above property. Since $\tilde{\sigma}$ is trivial on $B(\ell^2)$, so is $\phi_{\tilde{\sigma}}$. Then we can define the map $\phi_{\tilde{\rho}}$ on \widetilde{M} by $\phi_{\tilde{\sigma}} = \text{id} \otimes \phi_{\tilde{\rho}}$, which has the desired property. Hence we may and do assume that M is infinite.

Take an isometry $v \in (\text{id}, \bar{\rho}\rho)$. We set $\phi_{\tilde{\rho}}(x) := v^* \tilde{\rho}(x) v$ for $x \in \widetilde{M}$. By [17, Proposition 2.5 (1)], $v \in (\text{id}, \tilde{\rho}\tilde{\rho})$. Hence $\phi_{\tilde{\rho}}$ is a left inverse of $\tilde{\rho}$. By the previous lemma, we have

$$d(\bar{\rho}\rho)^{it} v^* [D\varphi \circ \phi_{\bar{\rho}\rho} : D\varphi]_t = \sigma_t^\varphi(v^*).$$

Using $\phi_{\bar{\rho}\rho} = \phi_\rho \phi_{\bar{\rho}}$ and (3.1), we have

$$\begin{aligned} \phi_{\tilde{\rho}}(x\lambda^\varphi(t)) &= v^* \tilde{\rho}(x\lambda^\varphi(t)) v = v^* \bar{\rho}(x) d(\rho)^{it} [D\varphi \circ \phi_{\bar{\rho}} : D\varphi]_t \lambda^\varphi(t) v \\ &= d(\rho)^{it} v^* \bar{\rho}(x) [D\varphi \circ \phi_{\bar{\rho}} : D\varphi]_t \sigma_t^\varphi(v) \lambda^\varphi(t) \\ &= d(\rho)^{it} v^* \bar{\rho}(x) [D\varphi \circ \phi_{\bar{\rho}} : D\varphi]_t \cdot d(\bar{\rho}\rho)^{-it} [D\varphi \circ \phi_{\bar{\rho}\rho} : D\varphi]_t^* v \lambda^\varphi(t) \\ &= d(\rho)^{-it} v^* \bar{\rho}(x) [D\varphi \circ \phi_{\bar{\rho}} : D\varphi \circ \phi_{\bar{\rho}\rho}]_t v \lambda^\varphi(t) \\ &= d(\rho)^{-it} v^* \bar{\rho}(x) \bar{\rho}([D\varphi : D\varphi \circ \phi_\rho]_t) v \lambda^\varphi(t) \\ &= d(\rho)^{-it} \phi_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t) \lambda^\varphi(t). \end{aligned}$$

\square

Hence the map $\phi_{\check{\rho}} := \phi_{\tilde{\rho}} \circ \theta_{-\log(d(\rho))}$ is a left inverse of $\check{\rho}$ such that

$$\phi_{\check{\rho}}(x\lambda^\varphi(t)) = \phi_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t) \lambda^\varphi(t).$$

Let $\varphi \in M_*$ be a faithful state. We identify \widetilde{M} with $M \rtimes_{\sigma^\varphi} \mathbb{R}$ via Π_φ . For $\rho \in \text{End}(M)_0$, we define the operator $U_{\check{\rho}}$ on $L^2(M) \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}, L^2(M))$ by

$$(U_{\check{\rho}}\xi)(s) = [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* V_\rho \xi(s) \quad \text{for all } \xi \in L^2(\mathbb{R}, L^2(M)), s \in \mathbb{R},$$

where V_ρ is an isometry defined as before by the fixed faithful normal state ψ .

Lemma 3.6. *For $\rho \in \text{End}(M)_0$, $U_{\check{\rho}}$ has the following properties:*

- (1) $U_{\check{\rho}}$ is an isometry.
- (2) $U_{\check{\rho}}x = \check{\rho}(x)U_{\check{\rho}}$ for all $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$.
- (3) $\phi_{\check{\rho}}(x) = U_{\check{\rho}}^* x U_{\check{\rho}}$ for all $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$.

Proof. (1) It is trivial.

(2) Let $x \in M$ and $t \in \mathbb{R}$. Then we have for $s \in \mathbb{R}$,

$$\begin{aligned} (U_{\check{\rho}}(\pi_{\sigma^\varphi}(x)\lambda(t))\xi)(s) &= [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* V_\rho ((\pi_{\sigma^\varphi}(x)\lambda(t))\xi)(s) \\ &= [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* V_\rho \sigma_{-s}^\varphi(x)\xi(-t+s) \\ &= [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* \rho(\sigma_{-s}^\varphi(x)) V_\rho \xi(-t+s) \\ &= [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* \sigma_{-s}^{\varphi \circ \phi_\rho}(\rho(x)) V_\rho \xi(-t+s) \\ &= \sigma_{-s}^\varphi(\rho(x)) [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* V_\rho \xi(-t+s) \\ &= \sigma_{-s}^\varphi(\rho(x)) [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* [D\varphi \circ \phi_\rho : D\varphi]_{t-s} \\ &\quad \cdot (U_{\check{\rho}}\xi)(-t+s) \\ &= \sigma_{-s}^\varphi(\rho(x)) \sigma_{-s}^\varphi([D\varphi \circ \phi_\rho : D\varphi]_t) \cdot (U_{\check{\rho}}\xi)(-t+s) \\ &= (\pi_{\sigma^\varphi}(\rho(x)) [D\varphi \circ \phi_\rho : D\varphi]_t) \lambda(t) U_{\check{\rho}}\xi(s) \\ &= (\check{\rho}(\pi_{\sigma^\varphi}(\rho(x))\lambda(t)) U_{\check{\rho}}\xi)(s). \end{aligned}$$

Hence the equality of (2) holds.

(3) By (2) and (3), we have for $x \in M$, $t \in \mathbb{R}$ and $\xi, \eta \in L^2(M) \otimes L^2(\mathbb{R})$,

$$\begin{aligned} &(U_{\check{\rho}}^* \pi_{\sigma^\varphi}(x)\lambda(t) U_{\check{\rho}}\xi, \eta) \\ &= (\pi_{\sigma^\varphi}(x)\lambda(t) U_{\check{\rho}}\xi, U_{\check{\rho}}\eta) \\ &= \int_{-\infty}^{\infty} ((\pi_{\sigma^\varphi}(x)\lambda(t) U_{\check{\rho}}\xi)(s), (U_{\check{\rho}}\eta)(s)) ds \\ &= \int_{-\infty}^{\infty} (\sigma_{-s}^\varphi(x) (U_{\check{\rho}}\xi)(-t+s), (U_{\check{\rho}}\eta)(s)) ds \\ &= \int_{-\infty}^{\infty} (\sigma_{-s}^\varphi(x) [D\varphi \circ \phi_\rho : D\varphi]_{t-s}^* V_\rho \xi(-t+s), [D\varphi \circ \phi_\rho : D\varphi]_{-s}^* V_\rho \eta(s)) ds \\ &= \int_{-\infty}^{\infty} ([D\varphi \circ \phi_\rho : D\varphi]_{-s} \sigma_{-s}^\varphi(x) [D\varphi \circ \phi_\rho : D\varphi]_{t-s}^* V_\rho \xi(-t+s), V_\rho \eta(s)) ds \\ &= \int_{-\infty}^{\infty} (\sigma_{-s}^{\varphi \circ \phi_\rho}(x) [D\varphi \circ \phi_\rho : D\varphi]_{-s} [D\varphi \circ \phi_\rho : D\varphi]_{t-s}^* V_\rho \xi(-t+s), V_\rho \eta(s)) ds \\ &= \int_{-\infty}^{\infty} (\sigma_{-s}^{\varphi \circ \phi_\rho}(x) \sigma_{-s}^{\varphi \circ \phi_\rho}([D\varphi \circ \phi_\rho : D\varphi]_t^*) V_\rho \xi(-t+s), V_\rho \eta(s)) ds \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} (V_{\rho}^* \sigma_{-s}^{\varphi \circ \phi_{\rho}}(x[D\varphi : D\varphi \circ \phi_{\rho}]_t) V_{\rho} \xi(-t+s), \eta(s)) ds \\
&= \int_{-\infty}^{\infty} (\phi_{\rho}(\sigma_{-s}^{\varphi \circ \phi_{\rho}}(x[D\varphi : D\varphi \circ \phi_{\rho}]_t)) \xi(-t+s), \eta(s)) ds \\
&= \int_{-\infty}^{\infty} (\sigma_{-s}^{\varphi}(\phi_{\rho}(x[D\varphi : D\varphi \circ \phi_{\rho}]_t)) \xi(-t+s), \eta(s)) ds \quad (\text{by (3.2)}) \\
&= (\pi_{\sigma^{\varphi}}(\phi_{\rho}(x[D\varphi : D\varphi \circ \phi_{\rho}]_t)) \lambda(t) \xi, \eta) \\
&= (\phi_{\check{\rho}}(\pi_{\sigma^{\varphi}}(x) \lambda(t)) \xi, \eta).
\end{aligned}$$

□

Lemma 3.7. *Assume that $(\rho^{\nu})_{\nu \in \mathbb{N}}$ converges to ρ in $\text{End}(M)_0$. Then $U_{\check{\rho}^{\nu}}$ converges to $U_{\check{\rho}}$ strongly.*

Proof. Since $U_{\check{\rho}^{\nu}}$ and $U_{\check{\rho}}$ are isometries, it suffices to show that $(U_{\check{\rho}^{\nu}} \xi, \eta)$ converges to $(U_{\check{\rho}} \xi, \eta)$ for all $\xi, \eta \in L^2(M) \otimes L^2(\mathbb{R})$. Moreover we may and do assume that ξ, η have compact supports on \mathbb{R} . Then we have

$$\begin{aligned}
(U_{\check{\rho}^{\nu}} \xi, \eta) &= \int_{-\infty}^{\infty} ((U_{\check{\rho}^{\nu}} \xi)(s), \eta(s)) ds \\
&= \int_{-\infty}^{\infty} ([D\varphi \circ \phi_{\rho^{\nu}} : D\varphi]_{-s}^* V_{\rho} \xi(s), \eta(s)) ds,
\end{aligned}$$

which converges to

$$\int_{-\infty}^{\infty} ([D\varphi \circ \phi_{\rho} : D\varphi]_{-s}^* V_{\rho} \xi(s), \eta(s)) ds$$

because for each $r > 0$, the cocycle $[D\varphi \circ \phi_{\rho^{\nu}} : D\varphi]_t$ uniformly converges to $[D\varphi \circ \phi_{\rho} : D\varphi]_t$ strongly as $\nu \rightarrow \infty$ for all $t \in [-r, r]$ [34, Theorem IX.1.19]. □

Theorem 3.8. *Let M be a factor. Then the normalized canonical extension is continuous, that is, if $(\rho^{\nu})_{\nu \in \mathbb{N}}$ converges to ρ in $\text{End}(M)_0$, then the pairs $\{(\check{\rho}^{\nu}, \phi_{\check{\rho}^{\nu}})\}_{\nu \in \mathbb{N}}$ converge to $(\check{\rho}, \phi_{\check{\rho}})$. In particular, $\check{\rho}^{\nu}(x)$ converges to $\check{\rho}(x)$ strongly* as $\nu \rightarrow \infty$ for all $x \in \widetilde{M}$.*

Proof. By the previous lemma, $U_{\check{\rho}^{\nu}}$ converges to $U_{\check{\rho}}$. Then the same proof as (3) \Rightarrow (1) and (1) \Rightarrow (2) of Lemma 3.3 works. □

3.3. Dominant weights and canonical extensions

In this subsection, we treat an infinite factor M . By the Takesaki duality, M is isomorphic to $\widetilde{M} \rtimes_{\theta} \mathbb{R}$ and \widetilde{M} is regarded as the centralizer of a dominant weight on M . We will study the canonical extension and the restriction of an endomorphism on the centralizer of a dominant weight.

Let φ be a dominant weight on M . Then the covariant system $\{M, \sigma^{\varphi}\}$ is dual, that is, there exists a one-parameter unitary group $\{v(t)\}_{t \in \mathbb{R}}$ in M such that $\sigma_s^{\varphi}(v(t)) = e^{-ist} v(t)$. Using the action $\theta_t := \text{Ad } v(t)$ on M_{φ} , we have the

natural isomorphism of the covariant systems $\{M, \sigma^\varphi\} \cong \{M_\varphi \rtimes_\theta \mathbb{R}, \hat{\theta}\}$. The dual action on \widetilde{M} is denoted by $\widehat{\sigma^\varphi}$ for a while.

Lemma 3.9. *Let $\rho \in \text{End}(M)_0$. Then there exists $u \in U(M)$ such that*

- (1) $(\varphi, \text{Ad } u \circ \rho)$ is an invariant pair in the sense of [17, Definition 2.2],
- (2) $u\rho(v(t))u^* = v(t)$.

Proof. By [18, Lemma 2.12 (ii)], we can take $u \in U(M)$ satisfying (1). Replace ρ with $\text{Ad } u \circ \rho$. Then $\sigma_t^\varphi \circ \rho = \rho \circ \sigma_t^\varphi$ by (3.1), and the unitary $w(t) := \rho(v(t))v(t)^*$ is contained in M_φ . Then $\{w(t)\}_t$ is a θ -cocycle. By using the stability of $\{M_\varphi, \theta\}$ [7, Theorem III.5.1 (ii)], there exists $\nu \in U(M_\varphi)$ such that $w(t) = \nu\theta_t(\nu^*)$. Then $(\varphi, \text{Ad } \nu^* \circ \rho)$ is an invariant pair, and $\nu^*\rho(v(t))\nu = v(t)$. \square

Replacing ρ with $\text{Ad } u \circ \rho$, we assume that ρ satisfies the conditions in the above lemma. Now we discuss how the canonical extension $\tilde{\rho} \in \text{End}(\widetilde{M})$ can be transformed to an endomorphism on M_φ . We let M act on a Hilbert space H . By the Takesaki duality, we have the isomorphism $\widetilde{M} \rightarrow M_\varphi \otimes B(L^2(\mathbb{R}))$ satisfying $xv(t)\lambda^\varphi(t) \mapsto \pi_\theta(x)(1 \otimes \mu_\theta(t))(1 \otimes \nu(t))$ for $x \in M_\varphi$ and $t \in \mathbb{R}$, where $\pi_\theta(x) \in M_\varphi \otimes L^\infty(\mathbb{R})$, $\mu_\theta(t) \in \mathbb{C} \otimes L(\mathbb{R})$ (the group von Neumann algebra of \mathbb{R}) and $\nu(t) \in \mathbb{C} \otimes L^\infty(\mathbb{R})$ are defined by

$$(\pi_\theta(x)\xi)(s) = \theta_{-s}(x)\xi(s), \quad (\mu_\theta(t)\xi)(s) = \xi(-t + s), \quad (\nu(t)\xi)(s) = e^{-its}\xi(s)$$

for all $\xi \in L^2(\mathbb{R}, H)$ and $s, t \in \mathbb{R}$.

Since $\tilde{\rho}(xv(t)\lambda^\varphi(t)) = \rho(x)v(t)\lambda^\varphi(t)$ and $(\rho \otimes \text{id}) \circ \pi_\theta = \pi_\theta \circ \rho$, $\tilde{\rho}$ is transformed to $\rho \otimes \text{id} \in \text{End}(M_\varphi \otimes B(L^2(\mathbb{R})))$ through the isomorphism. The dual action $\widehat{\sigma^\varphi}$ on \widetilde{M} is given by $\theta_t \otimes \text{Ad } \mu_\theta(t)$ on $M_\varphi \otimes B(L^2(\mathbb{R}))$. Hence we have the following isomorphism between the covariant systems with an endomorphism ρ :

$$\{\widetilde{M}, \widehat{\sigma^\varphi}, \tilde{\rho}\} \cong \{M_\varphi \otimes B(L^2(\mathbb{R})), \theta \otimes \text{Ad } \mu_\theta, \rho \otimes \text{id}\},$$

which means there exists an isomorphism $\Theta: \widetilde{M} \rightarrow M_\varphi \otimes B(L^2(\mathbb{R}))$ such that $\Theta \circ \widehat{\sigma^\varphi}_s = (\theta_s \otimes \text{Ad } \mu_\theta(s)) \circ \Theta$ and $\Theta \circ \tilde{\rho} = (\rho \otimes \text{id}) \circ \Theta$ for all $s \in \mathbb{R}$.

Lemma 3.10. *Let $M, \varphi, \{v(s)\}_{s \in \mathbb{R}}$ and θ be as before. Then for any $\rho \in \text{End}(M)_0$, there exists $u \in M$ and an isomorphism $\Psi_\rho: \widetilde{M} \rightarrow M_\varphi$ such that*

- (1) $(\varphi, \text{Ad } u \circ \rho)$ is an invariant pair,
- (2) $u\rho(v(s))u^* = v(s)$,
- (3) Ψ_ρ is an isomorphism between the following covariant systems:

$$\Psi_\rho: \{\widetilde{M}, \widehat{\sigma^\varphi}, \tilde{\rho}\} \rightarrow \{M_\varphi, \theta, \text{Ad } u \circ \rho|_{M_\varphi}\}.$$

Proof. We may assume that (φ, ρ) is an invariant pair and $\rho(v(s)) = v(s)$ for all $s \in \mathbb{R}$ as before. Since θ is a dual action [7, Theorem III.5.1 (ii)], $(M_\varphi)^\theta$ is isomorphic to M . Hence we can take an infinite dimensional Hilbert space $\mathcal{H} \subset (M_\varphi)^\theta$ with support 1. Let $\{\xi_i\}_{i=1}^\infty$ be an orthonormal basis of \mathcal{H} . Let $t_{\mathcal{H}}: \rho_{\mathcal{H}}(M_\varphi) \otimes B(\mathcal{H}) \rightarrow M_\varphi$ be the isomorphism such that $t_{\mathcal{H}}(\rho_{\mathcal{H}}(x) \otimes \xi_i \xi_j^*) = \rho_{\mathcal{H}}(x) \xi_i \xi_j^* = \xi_i x \xi_j^*$ for all $x \in M_\varphi$ and $i, j \in \mathbb{N}$. We define the unitary $u =$

$\sum_{i=1}^{\infty} \rho(\xi_i) \xi_i^*$. Then $u \in (M_\varphi)^\theta$ and $u\mathcal{H} = \rho(\mathcal{H})$. We also define the isomorphism $\Psi: B(L^2(\mathbb{R})) \rightarrow B(H)$ such that $\Psi(e_{ij}) = \xi_i \xi_j^*$, where $\{e_{ij}\}_{i,j=1}^{\infty}$ is a system of matrix units of $B(L^2(\mathbb{R}))$.

Now we introduce the isomorphism $\Phi: M_\varphi \otimes B(L^2(\mathbb{R})) \rightarrow M_\varphi$ defined by

$$\Phi: M_\varphi \otimes B(L^2(\mathbb{R})) \xrightarrow{\rho_{\mathcal{H}} \otimes \Psi} \rho_{\mathcal{H}}(M_\varphi) \otimes B(\mathcal{H}) \xrightarrow{t_{\mathcal{H}}} M_\varphi \xrightarrow{\text{Ad } u} M_\varphi.$$

Then we have $\Phi(x \otimes e_{ij}) = u \rho_{\mathcal{H}}(x) \xi_i \xi_j^* u^* = u \xi_i x \xi_j^* u^*$ for all $x \in M_\varphi$ and $i, j \in \mathbb{N}$. We will check that

$$\Phi \circ (\theta_s \otimes \text{id}) \circ \Phi^{-1} = \theta_s, \quad \Phi \circ (\rho \otimes \text{id}) \circ \Phi^{-1} = \text{Ad } \rho(u^*) \circ \rho = \rho \circ \text{Ad } u^*.$$

Indeed, for $x \in M_\varphi$ and $i, j \in \mathbb{N}$, we have

$$\begin{aligned} \Phi((\theta_s \otimes \text{id})(x \otimes e_{ij})) &= \Phi(\theta_s(x) \otimes e_{ij}) = u \xi_i \theta_s(x) \xi_j^* u^* = \theta_s(u \xi_i x \xi_j^* u^*) \\ &= \theta_s(\Phi(x \otimes e_{ij})), \end{aligned}$$

and

$$\begin{aligned} \Phi((\rho \otimes \text{id})(x \otimes e_{ij})) &= \Phi(\rho(x) \otimes e_{ij}) = u \xi_i \rho(x) \xi_j^* u^* = \rho(\xi_i x \xi_j^*) \\ &= \rho(u^*) \rho(u \xi_i x \xi_j^* u^*) \rho(u) = \rho(u^*) \rho(\Phi(x \otimes e_{ij})) \rho(u). \end{aligned}$$

Set $\mu(s) = \Phi(1 \otimes \mu_\theta(s))$. Since $1 \otimes \mu_\theta(s)$ is fixed by $\theta \otimes \text{id}$ and $\rho \otimes \text{id}$, so is $\mu(s)$ by θ and $\text{Ad } \rho(u^*) \circ \rho$. Hence we have the following isomorphism between the covariant systems with endomorphisms:

$$\Phi \circ \Theta: \{\widetilde{M}, \widehat{\sigma}^\varphi, \widetilde{\rho}\} \rightarrow \{M_\varphi, \theta \circ \text{Ad } \mu, \text{Ad } \rho(u^*) \circ \rho|_{M_\varphi}\}.$$

The one-parameter unitary group $\{\mu(s)\}_{s \in \mathbb{R}}$ is contained in M_φ^θ , and this is a θ -cocycle. By stability of θ [7, Theorem III.5.1 (ii)], there exists $w \in U(M_\varphi)$ such that $\mu(s) = w \theta_s(w^*)$ for all $s \in \mathbb{R}$. Then the equality $\text{Ad } \rho(u^*) \circ \rho(\mu(s)) = \mu(s)$ implies that $w^* \rho(u^* w) \in M_\varphi^\theta$. Indeed using $\rho(u) \in M_\varphi^\theta$ and $\rho \circ \theta_s = \theta_s \circ \rho$, we have

$$\begin{aligned} (w^* \rho(u^* w))^* \theta_s(w^* \rho(u^* w)) &= \rho(w^* u) w \theta_s(w^*) \theta_s(\rho(u^* w)) \\ &= \rho(w^*) \rho(u) \mu(s) \rho(u^*) \theta_s(\rho(w)) \\ &= \rho(w^*) \rho(\mu(s)) \theta_s(\rho(w)) \\ &= \rho(w^* \mu(s) \theta_s(w)) = 1. \end{aligned}$$

Now we have the following isomorphism:

$$\{\widetilde{M}, \widehat{\sigma}^\varphi, \widetilde{\rho}\} \xrightarrow{\Phi \circ \Theta} \{M_\varphi, \theta \circ \text{Ad } \mu, \text{Ad } \rho(u^*) \circ \rho|_{M_\varphi}\} \xrightarrow{\text{Ad } w^*} \{M_\varphi, \theta, \text{Ad}(w^* \rho(u^* w)) \circ \rho|_{M_\varphi}\}.$$

This is a desired one. Indeed, it is easy to see that $(\varphi, \text{Ad}(w^* \rho(u^* w)) \circ \rho)$ is an invariant pair since $w^* \rho(u^* w) \in M_\varphi^\theta$. Furthermore we have

$$\begin{aligned} \text{Ad}(w^* \rho(u^* w))(\rho(v(s))) &= w^* \rho(u^* w) v(s) \rho(w^* u) w \\ &= v(s) \theta_{-s}(w^* \rho(u^* w)) \rho(w^* u) w \\ &= v(s) w^* \rho(u^* w) \rho(w^* u) w = v(s). \end{aligned}$$

□

The following lemma is probably well-known by specialists, but we present a proof for readers' convenience.

Lemma 3.11. *Let M be an infinite factor and $\rho \in \text{End}(M)_0$. Then there exists an isomorphism $\pi: M \otimes B(\ell_2) \rightarrow M$ such that $[\rho] = [\pi \circ (\rho \otimes \text{id}) \circ \pi^{-1}]$ in $\text{Sect}(M)$.*

Proof. Take an infinite dimensional Hilbert space \mathcal{H} in M with support 1. Let $\{\xi_i\}_{i=1}^\infty$ be an orthonormal basis of \mathcal{H} . Define the isomorphism $t_{\mathcal{H}}: \rho_{\mathcal{H}}(M) \otimes B(\mathcal{H}) \rightarrow M$ by $t_{\mathcal{H}}(\rho_{\mathcal{H}}(x) \otimes \xi_i \xi_j^*) = \rho_{\mathcal{H}}(x) \xi_i \xi_j^* = \xi_i x \xi_j^*$ for $x \in M$ and $i, j \in \mathbb{N}$. Then the isomorphism $\pi: M \otimes B(\mathcal{H}) \rightarrow M$ is defined by $\pi = t_{\mathcal{H}} \circ (\rho_{\mathcal{H}} \otimes \text{id})$. Define the unitary $u = \sum_{i=1}^\infty \xi_i \rho(\xi_i^*)$. By direct computation, we see that $\pi \circ (\rho \otimes \text{id}) \circ \pi^{-1}(x) = \text{Ad } u \circ \rho(x)$ for all $x \in M$. Hence $[\pi \circ (\rho \otimes \text{id}) \circ \pi^{-1}] = [\rho] \in \text{Sect}(M)$. \square

3.4. Connes-Takesaki modules and approximately inner endomorphisms

Let M be a factor and $\rho \in \text{End}(M)_0$. Following [17, Definition 4.1], we say that ρ has a Connes-Takesaki module if the canonical extension $\tilde{\rho}$ satisfies $\tilde{\rho}(Z(\tilde{M})) = Z(\tilde{M})$. We denote by $\text{mod}(\rho)$ the restriction of $\tilde{\rho}$ to $Z(\tilde{M})$ and by $\text{End}(M)_{\text{CT}}$ the set of endomorphisms with Connes-Takesaki modules.

Let M be an infinite factor. Take a dominant weight φ on M and a one-parameter unitary group $\{v(t)\}_{t \in \mathbb{R}}$ such that $\sigma_s^\varphi(v(t)) = e^{-ist}v(t)$ as before. Set $\theta_s := \text{Ad } v(s)|_{M_\varphi} \in \text{Aut}(M_\varphi)$ as before. Taking Lemma 3.10 into account, we also denote by θ the dual action $\widehat{\sigma^\varphi}$ on \tilde{M} .

Lemma 3.12. *Let M be a type III_λ factor with $0 < \lambda < 1$. Let $\rho \in \text{End}(M)_{\text{CT}}$. Assume that there exists $s_0 \in \mathbb{R}$ such that $\text{mod}(\rho) = \theta_{s_0}$. Then for any generalized trace ψ on M , there exists a unitary $u \in M$ such that in the discrete decomposition $M = M_\psi \rtimes_\sigma \mathbb{Z}$,*

- (1) $\psi \circ \phi_{\text{Ad } u \circ \rho} = d(\rho)^{-1} e^{-s_0} \psi$,
- (2) $\text{Ad } u \circ \rho(U) = U$, where U is the unitary implementing σ .

Proof. By Lemma 3.10, we may assume that (φ, ρ) is an invariant pair, $\rho(v(s)) = v(s)$ for all $s \in \mathbb{R}$ and $\rho|_{Z(M_\varphi)} = \theta_{s_0}$. Since each generalized trace is unique up to scalar multiplications and inner perturbations, to achieve (1) for each generalized trace, it suffices to construct one generalized trace ψ satisfying (1).

We put $T = -2\pi/\log \lambda$. Then $\sigma_T^\varphi = \text{Ad } h^{iT}$ for some invertible $h \in Z(M_\varphi)_+$. Since $\sigma_T^\varphi(v(t)) = e^{-itT}v(t)$, we have $\theta_t(h^{-iT}) = e^{-itT}h^{-iT}$. We set $\psi := \varphi_{h^{-1}}$. Then it is well-known that ψ is a generalized trace. Indeed, it is trivial that ψ has the period T . So, we have to check $\psi(1) = \varphi(h^{-1}) = \infty$, but it is also trivial since φ is a dual weight. Set $w := v(s_0)^*$. Then $\text{Ad } w \circ \rho = \text{id}$ on $Z(M_\varphi)$, and $\text{Ad } w \circ \rho(h) = h$. Hence we have

$$\begin{aligned} \psi \circ \phi_{\text{Ad } w \circ \rho} &= \varphi_{h^{-1}} \circ \phi_{\text{Ad } w \circ \rho} = (\varphi \circ \phi_{\text{Ad } w \circ \rho})_{h^{-1}} \\ &= (\varphi \circ \phi_\rho \circ \text{Ad } w^*)_{h^{-1}} = d(\rho)^{-1} (\varphi \circ \text{Ad } v(s_0))_{h^{-1}} \\ &= d(\rho)^{-1} e^{-s_0} \varphi_{h^{-1}} = d(\rho)^{-1} e^{-s_0} \psi. \end{aligned}$$

Therefore, we may assume that ρ has the property $\psi \circ \phi_\rho = d(\rho)^{-1}e^{-s_0}\psi$. As is explained, we also may assume that ψ is a given generalized trace.

Now let $M = M_\psi \rtimes_\sigma \mathbb{Z}$ be the discrete decomposition with the implementing unitary U . Since σ^ψ and ρ commutes, we see that $\rho(U)U^*$ is in M_ψ . By stability of $\sigma = \text{Ad } U|_{M_\psi}$ [7, Theorem III.5.1 (i)], we can take a unitary $u \in M_\psi$ such that $\rho(U)U^* = u^*\sigma(u) = u^*UuU^*$. Hence we have $\text{Ad } u \circ \rho(U) = U$. Since $u \in M_\psi$, we have $\psi \circ \phi_{\text{Ad } u \circ \rho} = d(\rho)^{-1}e^{-s_0}\psi$. \square

Lemma 3.13. *Let M be a type III_0 factor. Let $\rho \in \text{End}(M)_{\text{CT}}$. Assume that there exists $s_0 \in \mathbb{R}$ such that $\text{mod}(\rho) = \theta_{s_0}$. Then there exists $\psi \in W_{\text{lac}}(M)$ and $u \in U(M)$ such that*

- (1) ψ has infinite multiplicity,
- (2) $\psi \circ \phi_{\text{Ad } u \circ \rho} = d(\rho)^{-1}e^{-s_0}\psi$,
- (3) $\text{Ad } u \circ \rho|_{Z(M_\psi)} = \text{id}$.

Proof. There exists $u \in U(M)$ satisfying Lemma 3.10. Replacing ρ with $\text{Ad } u \circ \rho$, we may assume that (φ, ρ) is an invariant pair, $\rho(v(s)) = v(s)$ and $\rho|_{Z(M_\varphi)} = \theta_{s_0}|_{Z(M_\varphi)}$. By perturbing ρ to $\text{Ad } v(s_0)^* \circ \rho$ again, we may and do assume that ρ satisfies $\varphi \circ \phi_\rho = d(\rho)^{-1}e^{-s_0}\varphi$, $\rho(v(s)) = v(s)$ and $\rho|_{Z(M_\varphi)} = \text{id}$.

Take $h \in Z(M_\varphi)_+$ such that a normal semifinite weight $\chi = \varphi_h$ is lacunary. Let $e \in Z(M_\varphi)$ be the support projection of h . Then χ is faithful on eMe . Since $\rho|_{Z(M_\varphi)} = \text{id}$, we have $\rho(h) = h$ and $\phi_\rho(h) = \phi_\rho(\rho(h)) = h$. Then

$$\chi \circ \phi_\rho = \varphi_h \circ \phi_\rho = (\varphi \circ \phi_\rho)_{\rho(h)} = d(\rho)^{-1}e^{-s_0}\varphi_h = d(\rho)^{-1}e^{-s_0}\chi.$$

Since $eM_\varphi e \subset M_\chi \subset eMe$ and $M'_\varphi \cap M = Z(M_\varphi)$ by Connes-Takesaki relative commutant theorem [7, Theorem II.5.1], we have $Z(M_\chi) \subset (eM_\varphi e)' \cap eMe = Z(M_\varphi)e$. Hence $\rho|_{Z(M_\chi)} = \text{id}$.

Take an isometry $w \in M$ such that $ww^* = e$. Then the map $\pi : M \ni x \mapsto wxw^* \in eMe$ is an isomorphism. We set $\chi' := \chi \circ \pi \in W_{\text{lac}}(M)$ and $\rho' := \pi^{-1} \circ \rho \circ \pi \in \text{End}(M)_0$. Then $\rho' = \text{id}$ on $Z(M_{\chi'})$. It is easy to see that $v := w^*\rho(w) \in M$ is unitary, and $\rho' = \text{Ad } v \circ \rho$. Hence $\phi_{\rho'} = \phi_\rho \circ \text{Ad } v^*$ and

$$\begin{aligned} \chi' \circ \phi_{\rho'} &= (\chi \circ \text{Ad } w) \circ (\phi_\rho \circ \text{Ad } v^*) = \chi \circ \phi_\rho \circ \text{Ad } \rho(w)\rho(w^*)w \\ &= d(\rho)^{-1}e^{-s_0}\chi \circ \text{Ad } \rho(e)w = d(\rho)^{-1}e^{-s_0}\chi \circ \text{Ad } w = d(\rho)^{-1}e^{-s_0}\chi'. \end{aligned}$$

Note that χ' may not have infinite multiplicity. Consider the weight $\psi' := \chi' \otimes \text{Tr}$ and the endomorphism $\rho' \otimes \text{id}$ on $M \otimes B(\ell_2)$. Then by Lemma 3.11, there exists an isomorphism $\pi' : M \otimes B(\ell_2) \rightarrow M$ and a unitary $u \in U(M)$ such that $\text{Ad } u \circ \rho' = \pi' \circ (\rho' \otimes \text{id}) \circ \pi'^{-1}$. Set $\psi := (\chi' \otimes \text{Tr}) \circ \pi'^{-1}$. Then this ψ and $u \in U(M)$ are desired ones. \square

Let $M = M_\psi \rtimes_\sigma \mathbb{Z}$ be the discrete decomposition of M with the implementing unitary U . Then the same proof of [19, Lemma 2] works in our case, and we can take a unitary $v \in U(M_\psi)$ with $\text{Ad } vu \circ \rho(U) = U$. Hence the following holds.

Lemma 3.14. *Let M be a factor of type III_0 . Let $\rho \in \text{End}(M)_0$. Suppose that $\text{mod}(\rho) = \theta_{s_0}$ for some $s_0 \in \mathbb{R}$. Then there exist $\psi \in W_{\text{lac}}(M)$ with infinite multiplicity and $u \in U(M)$ such that*

- (1) *In the discrete decomposition $M = M_\psi \vee \{U\}''$, we have $\text{Ad } u \circ \rho|_{Z(M_\psi)} = \text{id}$,*
- (2) *$\psi \circ \phi_{\text{Ad } u \circ \rho} = d(\rho)^{-1} e^{-s_0} \psi$,*
- (3) *$\text{Ad } u \circ \rho(U) = U$.*

Now we prove the main result of this section.

Theorem 3.15. *Let M be a hyperfinite factor. Let $\rho \in \text{End}(M)_0$ and $r > 0$. Then the following conditions are equivalent:*

- (1) $\rho \in \overline{\text{Int}}_r(M)$,
- (2) $\rho \in \text{End}(M)_{\text{CT}}$ and $\text{mod}(\rho) = \theta_{\log(r/d(\rho))}$.

• *Proof of (2) \Rightarrow (1) in Theorem 3.15 for type I factors.*

Assume that $\rho \in \text{End}(M)_{\text{CT}} = \text{End}(M)$ has the Connes-Takesaki module $\text{mod}(\rho) = \theta_{\log(r/d(\rho))}$. Since any endomorphism on M is inner, $\rho \in \text{Int}_{d(\rho)}(M)$ and $\theta_{\log(r/d(\rho))} = \text{id}$. The space of the flow of weights of a type I factor is isomorphic to \mathbb{R} with the additive translation flow. Hence $\log(r/d(\rho)) = 0$, and $r = d(\rho)$. \square

• *Proof of (2) \Rightarrow (1) in Theorem 3.15 for the hyperfinite type II_1 factor.*

Let $\tau \in M_*$ be the tracial state. Then the canonical core $\widetilde{M} = M \rtimes_{\sigma^\tau} \mathbb{R}$ is naturally regarded as $M \otimes \{\lambda^\tau(t)\}_{t \in \mathbb{R}}''$, and $Z(\widetilde{M}) = \{\lambda^\tau(t)\}_{t \in \mathbb{R}}''$. Hence ρ has the Connes-Takesaki module with $\text{mod}(\rho) = \theta_{\log(r/d(\rho))}$ if and only if $d(\rho)^{it} [D\tau \circ \phi_\rho : D\tau]_t = d(\rho)^{it} r^{-it}$ for all $t \in \mathbb{R}$. This implies $\tau \circ \phi_\rho = r^{-1} \tau$, but $\tau(1) = 1$ yields that r must be equal to 1. Then $\tau \circ \phi_\rho = \tau$. Since ρ preserves the tracial state τ , we see that ρ is approximately inner of rank 1 by Lemma 2.14. \square

• *Proof of (2) \Rightarrow (1) in Theorem 3.15 for the hyperfinite type II_∞ factor.*

Let τ be the normal semifinite tracial weight on M . Then $Z(\widetilde{M}) = \{\lambda^\tau(t)\}_{t \in \mathbb{R}}''$, and $\tau \circ \phi_\rho = r^{-1} \tau$ holds. Then Lemma 2.15 implies that $\rho \in \overline{\text{Int}}_r(M)$. \square

• *Proof of (2) \Rightarrow (1) in Theorem 3.15 for the hyperfinite type III_1 factor.*

We make use of Popa's result on approximate innerness of hyperfinite subfactors of type III_1 . Since the flow of weights is trivial, $\text{End}(M)_0 = \text{End}(M)_{\text{CT}}$ and the modules of endomorphisms are trivial. Hence we have to prove $\text{End}(M)_0 = \overline{\text{Int}}_r(M)$ for all $r > 0$. Let $\rho \in \text{End}(M)_0$. Take any $\mu_0, \mu_1 > 0$ with $\mu_0 + \mu_1 = 1$. Set $r := \mu_1/\mu_0$. Consider the locally trivial subfactor $N^{(\text{id}, \rho)} \subset M^{(\text{id}, \rho)}$ with the expectation $E^{(\mu_0, \mu_1)}$. Then the subfactor is approximately inner by [29, Theorem 2.9 (iv)]. We note that that Popa's result states for minimal expectations, but the same proof is applicable for general expectations because we can prove that the Jones projections given in his proof are contained in the centralizer of the given state. Hence $\rho \in \overline{\text{Int}}_r(M)$ by Theorem 2.9. \square

• *Proof of (2) \Rightarrow (1) in Theorem 3.15 for hyperfinite type III_0 factors.*

We make use of the discrete decomposition of M to reduce the problem to that of a type II von Neumann algebra. By Lemma 3.14 for $s_0 = \log(rd(\rho)^{-1})$, after

perturbing ρ by an inner automorphism, we may assume that there exists a lacunary weight ψ on M with infinite multiplicity such that $\psi \circ \phi_\rho = d(\rho)^{-1} e^{-s_0} \psi = r^{-1} \psi$, $\rho|_{Z(M_\psi)} = \text{id}$ and $\rho(U) = U$, where U is the implementing unitary in the discrete decomposition $M = M_\psi \rtimes_\sigma \mathbb{Z}$. Since $\phi_\rho: M \rightarrow M$ is the standard left inverse, we have $\text{Ad } U \circ \phi_\rho = \phi_\rho \circ \text{Ad } U$ on M by uniqueness.

Set $\tau := \psi|_{M_\psi}$, which is a faithful normal semifinite trace on the type II_∞ von Neumann algebra $N := M_\psi$. Lemma 2.16 shows that $\rho|_N$ is approximately inner of rank r with respect to $\phi_\rho|_N$. Hence there exist partial isometries $\{v_i^\nu\}_{i=1}^{[r]+1}$, $\nu \in \mathbb{N}$, in N such that $(v_i^\nu)^* v_j^\nu = \delta_{i,j} 1$ for $1 \leq i, j \leq [r]$, $\sum_{i=1}^{[r]+1} v_i^\nu (v_i^\nu)^* = 1$ and

$$\lim_{\nu \rightarrow \infty} \|r^{-1} v_i^\nu \cdot \chi - \chi \circ \phi_\rho \cdot v_i^\nu\|_{N_*} = 0 \quad \text{for all } \chi \in N_*. \quad (3.4)$$

Since $\sigma \circ \phi_\rho = \phi_\rho \circ \sigma$ on N , we also have

$$\lim_{\nu \rightarrow \infty} \|r^{-1} \sigma(v_i^\nu) \cdot \chi - \chi \circ \phi_\rho \cdot \sigma(v_i^\nu)\|_{N_*} = 0 \quad \text{for all } \chi \in N_*.$$

This implies that $((v_i^\nu)^* \sigma(v_j^\nu))_\nu$ and $((v_i^\nu)^* v_j^\nu)_\nu$ are central sequences in N . Recall the quotient map $\pi_\omega: \mathcal{N}(\mathcal{T}_\omega) \rightarrow N^\omega$. We set the following elements:

$$v_i := \pi_\omega((v_i^\nu)_\nu), \quad w_{ij} := v_i^* \sigma^\omega(v_j), \quad p_i := v_i^* v_i.$$

Then we have $v_i^* v_j = \delta_{ij} p_i$. Moreover, w_{ij} is in N_ω , and $p_i = 1$ for $1 \leq i \leq [r]$ and $p_{[r]+1}$ is a projection in N_ω . On w_{ij} , we have the following relations:

$$\sum_{j=1}^{[r]+1} w_{ij} w_{kj}^* = \delta_{ik} p_i, \quad \sum_{j=1}^{[r]+1} w_{jk}^* w_{ji} = \delta_{ik} \sigma_\omega(p_i) \quad \text{for all } 1 \leq i, k \leq [r] + 1, \quad (3.5)$$

and

$$\sigma^\omega(v_j) = \sum_{i=1}^{[r]+1} v_i w_{ij}. \quad (3.6)$$

We will prove the following two claims.

Claim 1. We can replace a sequence $(v_{[r]+1}^\nu)_\nu$ so that $\sigma_\omega(p_{[r]+1}) = p_{[r]+1}$.

(Proof of Claim 1.) Consider the von Neumann algebra $N_\omega \otimes B(\mathbb{C}^{[r]+1})$. We set $w := \sum_{i,j=1}^{[r]+1} w_{ij} \otimes e_{ij}$ and $p := \sum_{i=1}^{[r]+1} p_i \otimes e_{ii}$, where $\{e_{ij}\}_{i,j=1}^{[r]+1}$ is a system of matrix units of $B(\mathbb{C}^{[r]+1})$. The equality (3.5) yields

$$ww^* = p, \quad w^* w = (\sigma_\omega \otimes \text{id})(p).$$

In particular, p and $(\sigma_\omega \otimes \text{id})(p)$ are equivalent in $N_\omega \otimes B(\mathbb{C}^{[r]+1})$. Since $N_\omega \otimes B(\mathbb{C}^{[r]+1})$ is finite, $1 - p = p_{[r]+1} \otimes e_{[r]+1, [r]+1}$ and $1 - (\sigma_\omega \otimes \text{id})(p) = \sigma_\omega(p_{[r]+1}) \otimes e_{[r]+1, [r]+1}$ are also equivalent in $N_\omega \otimes B(\mathbb{C}^{[r]+1})$. Hence $p_{[r]+1}$ and $\sigma_\omega(p_{[r]+1})$ are equivalent in N_ω . Take a unitary $v \in N_\omega$ such that $\sigma_\omega(p_{[r]+1}) = v^* p_{[r]+1} v$.

We note that the \mathbb{Z} -action σ_ω on N_ω is stable [19, Lemma 4]. Hence we can take a unitary $u \in N_\omega$ such that $v = u \sigma_\omega(u^*)$. Then we have $\sigma_\omega(u^* p_{[r]+1} u) = u^* p_{[r]+1} u$. Let $(u^\nu)_\nu$ be a representing sequence of u such that u^ν is a unitary for all $\nu \in \mathbb{N}$. When we replace $v_{[r]+1}^\nu$ with $v_{[r]+1}^\nu u^\nu$ and choose a subsequence of $(v_{[r]+1}^\nu u^\nu)_\nu$, we

see that the new family $\{v_i^\nu\}_{i=1}^{[r]} \cup \{v_{[r]+1}^\nu u^\nu\}$ also satisfies the above conditions (1), (2), (3) and also $\sigma_\omega(p_{[r]+1}) = p_{[r]+1}$. \square

By using Claim 1, we assume that $\sigma_\omega(p_{[r]+1}) = p_{[r]+1}$.

Claim 2. We can replace the sequences $(v_i^\nu)_\nu$ for $1 \leq i \leq [r] + 1$ so that $\sigma(v_i) - v_i \rightarrow 0$ strongly* as $\nu \rightarrow \infty$ for all $1 \leq i \leq [r] + 1$.

(Proof of Claim 2.) Since $ww^* = p = (\sigma_\omega \otimes \text{id})(p) = w^*w$, w is a unitary in $p(N_\omega \otimes B(\mathbb{C}^{[r]+1}))p$. By our assumption, we can consider the reduced \mathbb{Z} -action $(\sigma_\omega \otimes \text{id})^p$. It is easy to see that $(\sigma_\omega \otimes \text{id})^p$ also has stability by using Rohlin towers in $Z(N)$ as in the proof of [19, Lemma 3, 4]. Hence there exists a unitary $\mu \in p(N_\omega \otimes B(\mathbb{C}^{[r]+1}))p$ such that

$$w = \mu(\sigma_\omega \otimes \text{id})(\mu^*). \quad (3.7)$$

Now we set

$$\bar{v}_i := \sum_{j=1}^{[r]+1} v_j \mu_{ji} \in N^\omega, \quad (3.8)$$

where μ_{ji} is the (j, i) -entry of μ . Then we have

$$\begin{aligned} (\bar{v}_i)^* \bar{v}_j &= \sum_{k, \ell=1}^{[r]+1} \mu_{ki}^* v_k^* v_\ell \mu_{\ell j} = \sum_{k=1}^{[r]+1} \mu_{ki}^* p_k \mu_{kj} \\ &= (\mu^* p \mu)_{ij} = (\mu^* \mu)_{ij} = \delta_{ij} p_i. \end{aligned}$$

Using (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned} \sigma^\omega(\bar{v}_i) &= \sum_{j=1}^{[r]+1} \sigma^\omega(v_j) \sigma_\omega(\mu_{ji}) = \sum_{j,k=1}^{[r]+1} v_k w_{kj} \sigma_\omega(\mu_{ji}) \\ &= \sum_{k=1}^{[r]+1} v_k \mu_{ki} = \bar{v}_i. \end{aligned} \quad (3.9)$$

Now we take a representing sequence $(\mu^\nu)_\nu \in N \otimes B(\mathbb{C}^{[r]+1})$ of μ such that

$$(\mu^\nu)^* \mu^\nu = \sum_{i=1}^{[r]} (1 \otimes e_{ii}) + (v_{[r]+1}^\nu)^* (v_{[r]+1}^\nu) \otimes e_{[r]+1, [r]+1} = \mu^\nu (\mu^\nu)^*.$$

Using $(\mu^\nu)_\nu$, we take a representing sequence $(\bar{v}_i^\nu)_\nu$ of \bar{v}_i defined by

$$\bar{v}_i^\nu := \sum_{j=1}^{[r]+1} v_j^\nu \mu_{ji}^\nu$$

for all $1 \leq i \leq [r] + 1$ and $\nu \in \mathbb{N}$. Then we have

$$(\bar{v}_i^\nu)^* \bar{v}_j^\nu = \sum_{k, \ell=1}^{[r]+1} (\mu_{ki}^\nu)^* (v_k^\nu)^* v_\ell^\nu \mu_{\ell j}^\nu = \sum_{k=1}^{[r]+1} (\mu_{ki}^\nu)^* \mu_{kj}^\nu = \delta_{ij} (v_i^\nu)^* v_j^\nu,$$

and for $\chi \in N_*$,

$$\begin{aligned}
\lim_{\nu \rightarrow \omega} \|r^{-1}\bar{v}_i^\nu \cdot \chi - \chi \circ \phi_\rho \cdot \bar{v}_i^\nu\|_{N_*} &\leq \lim_{\nu \rightarrow \omega} \sum_{j=1}^{[r]+1} \|r^{-1}(v_j^\nu \mu_{ji}^\nu) \cdot \chi - \chi \circ \phi_\rho \cdot (v_j^\nu \mu_{ji}^\nu)\|_{N_*} \\
&= \lim_{\nu \rightarrow \omega} \sum_{j=1}^{[r]+1} \|r^{-1}v_j^\nu \cdot \chi \cdot \mu_{ji}^\nu - \chi \circ \phi_\rho \cdot (v_j^\nu \mu_{ji}^\nu)\|_{N_*} \\
&\leq \lim_{\nu \rightarrow \omega} \sum_{j=1}^{[r]+1} \|r^{-1}v_j^\nu \cdot \chi - \chi \circ \phi_\rho \cdot v_j^\nu\|_{N_*} = 0,
\end{aligned} \tag{3.10}$$

where we have used (3.4) and $\pi_\omega((\mu_{ji}^\nu)_\nu) = \mu_{ji} \in N_\omega$. Moreover $\sigma^\omega(\bar{v}_i) = \bar{v}_i$ shows the following strong* convergence:

$$\lim_{\nu \rightarrow \omega} \sigma(\bar{v}_i^\nu) - \bar{v}_i^\nu = 0. \tag{3.11}$$

Therefore there exists a subsequence of $(\bar{v}_i^\nu)_\nu$ such that the above limits (3.10) and (3.11) are taken by $\nu \rightarrow \infty$. This is a desired one in Claim 2. \square

Let us take $(v_i^\nu)_\nu$ satisfying Claim 2. We show that

$$\lim_{\nu \rightarrow \infty} \|r^{-1}v_i^\nu \cdot \chi - \chi \circ \phi_\rho \cdot v_i^\nu\|_{M_*} = 0 \quad \text{for all } 1 \leq i \leq [r] + 1, \chi \in M_*. \tag{3.12}$$

This implies that $\rho \in \overline{\text{Int}}_r(M)$.

For $k \in \mathbb{Z}$, set a bounded linear map \mathcal{E}_k on M such that $\mathcal{E}_k(xU^\ell) = \delta_{k\ell}xU^\ell$ for all $x \in N$ and $\ell \in \mathbb{Z}$. Since $\{\chi \circ \mathcal{E}_k \mid \chi \in M_*, k \in \mathbb{Z}\}$ is total in M_* , it suffices to prove (3.12) for $\chi \circ \mathcal{E}_k$, $\chi \in M_*$. Let $x \in M$ and $x = \sum_{k \in \mathbb{Z}} x_k U^k$ be the (formal) expansion of x . Then

$$\begin{aligned}
(r^{-1}v_i^\nu \cdot \chi \circ \mathcal{E}_k - \chi \circ \mathcal{E}_k \circ \phi_\rho \cdot v_i^\nu)(x) &= r^{-1}\chi(\mathcal{E}_k(xv_i^\nu)) - \chi(\mathcal{E}_k(\phi_\rho(v_i^\nu x))) \\
&= r^{-1}\chi(x_k \sigma^k(v_i^\nu) U^k) - \chi(\phi_\rho(v_i^\nu x_k) U^k) \\
&= (r^{-1}\sigma^k(v_i^\nu) \cdot (U^k \chi) - (U^k \chi) \circ \phi_\rho \cdot v_i^\nu)(x_k).
\end{aligned}$$

Since $\|x_k\| \leq \|x\|$, we have

$$\|r^{-1}v_i^\nu \cdot \chi \circ \mathcal{E}_k - \chi \circ \mathcal{E}_k \circ \phi_\rho \cdot v_i^\nu\|_{M_*} \leq \|r^{-1}\sigma^k(v_i^\nu) \cdot (U^k \chi)|_N - (U^k \chi)|_N \circ \phi_\rho \cdot v_i^\nu\|_{N_*}.$$

Since $\sigma^k(v_i^\nu) - v_i^\nu \rightarrow 0$ strongly* as $\nu \rightarrow \infty$, we have

$$\begin{aligned}
&\overline{\lim}_{\nu \rightarrow \infty} \|r^{-1}v_i^\nu \cdot \chi \circ \mathcal{E}_k - \chi \circ \mathcal{E}_k \circ \phi_\rho \cdot v_i^\nu\|_{M_*} \\
&\leq \overline{\lim}_{\nu \rightarrow \infty} \|r^{-1}\sigma^k(v_i^\nu) \cdot (U^k \chi)|_N - (U^k \chi)|_N \circ \phi_\rho \cdot v_i^\nu\|_{N_*} \\
&\leq \overline{\lim}_{\nu \rightarrow \infty} \|r^{-1}\sigma^k(v_i^\nu) \cdot (U^k \chi)|_N - r^{-1}v_i^\nu \cdot (U^k \chi)|_N\|_{N_*} \\
&\quad + \overline{\lim}_{\nu \rightarrow \infty} \|r^{-1}v_i^\nu \cdot (U^k \chi)|_N - (U^k \chi)|_N \circ \phi_\rho \cdot v_i^\nu\|_{N_*} = 0.
\end{aligned}$$

This shows (3.12). \square

• *Proof of (2) \Rightarrow (1) in Theorem 3.15 for the hyperfinite type III_λ factor with $0 < \lambda < 1$.*

We prove $\rho \in \overline{\text{Int}}_r(M)$ along with the proof of type III_0 case by using Lemma 3.12. Let ψ be a generalized trace and $M = M_\psi \rtimes_\sigma \mathbb{Z}$ the discrete decomposition. Then the \mathbb{Z} -action σ_ω on $(M_\psi)_\omega$ is stable [4, Theorem 2.1.3]. Here we note that σ is centrally free, i.e., $\sigma^n \in \text{Cnt}(M_\psi)$ if and only if $n = 0$, which follows from the fact that $\text{Cnt}(M_\psi) = \text{Int}(M_\psi)$ [4, Lemma 5]. So, Claim 1 still holds. Since any reduction of outer automorphism on a factor is still outer, Claim 2 also holds. Thus we can prove that $\rho \in \overline{\text{Int}}_r(M)$ as in type III_0 case. \square

Therefore we have proved the implication (2) \Rightarrow (1) in Theorem 3.15.

• *Proof of (1) \Rightarrow (2) in Theorem 3.15.*

First we assume $r \in \mathbb{N}$. Take r -dimensional Hilbert spaces $\mathcal{H}^\nu \subset M$ with support 1 for $\nu \in \mathbb{N}$, such that $\rho_{\mathcal{H}^\nu}$ converges to ρ as $\nu \rightarrow \infty$. Recall that the normalized canonical extension is continuous by Theorem 3.8. Hence $\rho_{\mathcal{H}^\nu} \rightarrow \rho$ implies $\theta_{\log(d(\rho_{\mathcal{H}^\nu}))}(\tilde{\rho}_{\mathcal{H}^\nu}(x)) \rightarrow \theta_{\log(d(\rho))}(\tilde{\rho}(x))$ strongly* as $\nu \rightarrow \infty$ for all $x \in \widetilde{M}$. Using $d(\rho_{\mathcal{H}^\nu}) = r$ and $\tilde{\rho}_{\mathcal{H}^\nu}|_{Z(\widetilde{M})} = \text{id}$, we have $\theta_{\log(d(\rho))}\tilde{\rho}|_{Z(\widetilde{M})} = \theta_{\log r}$. This shows that ρ has the Connes-Takesaki module $\text{mod}(\rho) = \theta_{\log(r/d(\rho))}$.

Next we treat a general case. Let $r > 0$. Take $s \in \mathbb{R}$ such that $e^s r \in \mathbb{N}$. Since $\text{mod}: \text{Aut}(M) \rightarrow \text{Aut}_\theta(Z(\widetilde{M}))$ is surjective [13, 32], there exists $\alpha \in \text{Aut}(M)$ such that $\tilde{\alpha} = \theta_s$ on $Z(\widetilde{M})$. By using (2) \Rightarrow (1) of Theorem 3.15, which we have already proved, we see that $\alpha \in \overline{\text{Int}}_{e^s}(M)$. Then $\alpha \circ \rho \in \overline{\text{Int}}_{e^s r}(M)$ by Lemma 2.12. Since $e^s r \in \mathbb{N}$, $\alpha \rho$ has the Connes-Takesaki module and $\text{mod}(\alpha \rho) = \theta_{\log(e^s r/d(\rho))}$ on $Z(\widetilde{M})$. This also shows that ρ has a Connes-Takesaki module, and

$$\begin{aligned} \text{mod}(\rho) &= \text{mod}(\alpha^{-1}) \text{mod}(\alpha \rho) = \tilde{\alpha}^{-1} \circ \theta_{\log(e^s r/d(\rho))} = \theta_{-s} \circ \theta_{\log(e^s r/d(\rho))} \\ &= \theta_{\log(r/d(\rho))}. \end{aligned}$$

\square

Corollary 3.16. *Let M be a hyperfinite factor and $r, s > 0$.*

(1) *When M is of type I, then*

$$\text{End}(M)_0 = \bigcup_{n \in \mathbb{N}} \text{Int}_n(M).$$

(2) *When M is of type II_1 with the tracial state τ , then*

$$\begin{aligned} \overline{\text{Int}}_1(M) &= \{\rho \in \text{End}(M)_0 \mid \tau \circ \phi_\rho = \tau\}, \\ \overline{\text{Int}}_r(M) &= \emptyset \quad \text{if } r \neq 1. \end{aligned}$$

(3) *When M is of type II_∞ with the trace τ , then*

$$\overline{\text{Int}}_r(M) = \{\rho \in \text{End}(M)_0 \mid \tau \circ \phi_\rho = r^{-1}\tau\}.$$

(4) *When M is of type III_1 , then*

$$\text{End}(M)_0 = \overline{\text{Int}}_r(M).$$

(5) When M is of type III_λ with $0 < \lambda < 1$ with the generalized trace φ , then

$$\begin{aligned}\overline{\text{Int}}_r(M) &= \{\rho \in \text{End}(M)_0 \mid \varphi \circ \phi_\rho = r^{-1}\varphi \circ \text{Ad } u \text{ for some } u \in U(M)\} \\ &= \overline{\text{Int}}_{r\lambda^n}(M) \text{ for all } n \in \mathbb{Z},\end{aligned}$$

$$\overline{\text{Int}}_r(M) \cap \overline{\text{Int}}_s(M) = \emptyset \text{ if } s \neq r\lambda^n \text{ for all } n \in \mathbb{Z}$$

(6) When M is of type III_0 ,

$$\overline{\text{Int}}_r(M) \cap \overline{\text{Int}}_s(M) = \emptyset \text{ if } r \neq s.$$

In the next section, we study a relation between centrally trivial endomorphisms and modular endomorphisms [17, Definition 3.1]. Note that the statistical dimension of a modular endomorphism is an integer.

Corollary 3.17. *Let M be a hyperfinite factor and ρ a modular endomorphism on M . Then $\rho \in \overline{\text{Int}}_{d(\rho)}(M)$.*

Proof. Since $\tilde{\rho}$ is inner, $\text{mod}(\rho) = \text{id} = \theta_{\log(d(\rho)/d(\rho))}$. By Theorem 3.15, we see that $\rho \in \overline{\text{Int}}_{d(\rho)}(M)$. \square

For conjugation of an endomorphism, a rank of approximate innerness behaves as follows. It seems that the assumption on hyperfiniteness is unnecessary, but we have no proof so far.

Corollary 3.18 (Conjugation rule). *Let M be a hyperfinite infinite factor and $\rho \in \overline{\text{Int}}_r(M)$ for some $r > 0$. Then $\bar{\rho} \in \overline{\text{Int}}_{r^{-1}d(\rho)^2}(M)$. In particular, $\rho\bar{\rho} \in \overline{\text{Int}}_{d(\rho)^2}(M)$.*

Proof. By Theorem 3.15, we have $\text{mod}(\rho) = \theta_{\log(r/d(\rho))}$, and

$$\text{mod}(\bar{\rho}) = \text{mod}(\rho)^{-1} = \theta_{-\log(r/d(\rho))} = \theta_{\log(d(\rho)/r)} = \theta_{\log(d(\rho)^2/rd(\bar{\rho}))}.$$

Hence $\bar{\rho} \in \overline{\text{Int}}_{d(\rho)^2/r}(M)$ again by Theorem 3.15. Then Lemma 2.12 implies that $\rho\bar{\rho} \in \overline{\text{Int}}_{d(\rho)^2}(M)$. \square

4. CENTRALLY TRIVIAL ENDOMORPHISMS

4.1. Centrally trivial endomorphisms

Let M be a factor and $\rho \in \text{End}(M)_0$. As is shown in [24, Lemma 3.3], we can define $\rho^\omega \in \text{End}(M^\omega)$ by

$$\rho^\omega(\pi_\omega((x^\nu)_\nu)) = \pi_\omega((\rho(x^\nu))_\nu) \quad \text{for all } (x^\nu)_\nu \in \mathcal{N}(\mathcal{T}_\omega(M)).$$

The map $\text{End}(M)_0 \ni \rho \mapsto \rho^\omega \in \text{End}(M^\omega)$ is a semigroup homomorphism.

Definition 4.1. Let M be a factor and $\rho \in \text{End}(M)_0$. We say that ρ is *centrally trivial* if and only if $\rho^\omega = \text{id}$ on M_ω .

We only consider centrally trivial endomorphisms with finite index. However, we should mention that it can be defined for an endomorphism with infinite index if this has a left inverse. We denote by $\text{Cnd}(M)$ the set of centrally trivial

endomorphisms on M . Since $\text{Int}(M) \subset \text{Cnd}(M)$, the central triviality is a property for sectors. We show that the set $\text{Cnd}(M)$ is closed under the composition, decomposition, direct sum and conjugation when M is infinite.

Definition 4.2. Let $N \subset M$ be an inclusion of factors.

- (1) $\mathcal{C}_\omega(M, N) := \{(x^\nu)_\nu \in \ell^\infty(\mathbf{N}, N) \mid \lim_{\nu \rightarrow \omega} \|[\varphi, x^\nu]\| = 0, \varphi \in M_*\}$.
- (2) $C_\omega(M, N) := \mathcal{C}_\omega(M, N)/\mathcal{T}_\omega(N)$.

If $N = M$, then it is obvious that $C_\omega(M, M) = M_\omega$.

Lemma 4.3. Let $N \subset M$ be an inclusion of factors with finite index. Let $M \supset N \supset N_1$ be a downward basic construction with respect to the minimal expectation $E: M \rightarrow N$. Then $C_\omega(M, N) = C_\omega(M, N_1)$.

Proof. Let $(x^\nu)_\nu \in \mathcal{C}_\omega(M, N)$, E_1 be the minimal expectation from N onto N_1 , and $e \in M$ the Jones projection for $N_1 \subset N$. Then $ex^\nu e - x^\nu e = [e, x^\nu]e$ converges to 0 σ -strongly*. Since $ex^\nu e = E_1(x^\nu)e$ and $E(e) = [M : N]_0^{-1}$, $(E_1(x^\nu) - x^\nu)_\nu$ converges to 0 strongly*. Hence any element in $C_\omega(M, N)$ is represented by an element in $\mathcal{C}_\omega(M, N_1)$. \square

Lemma 4.4. Let $\rho \in \text{Cnd}(M)$. Then $C_\omega(M, \rho(M)) = M_\omega$ holds.

Proof. Let $(\rho(x^\nu))_\nu \in \mathcal{C}_\omega(M, \rho(M))$. Since ϕ_ρ preserves M_ω , $(x^\nu)_\nu = (\phi_\rho(\rho(x^\nu)))_\nu$ is an ω -centralizing sequence of M . The central triviality of ρ implies that $(\rho(x^\nu))_\nu$ and $(x^\nu)_\nu$ are equivalent. Hence $C_\omega(M, \rho(M)) \subset M_\omega$ holds. Since $M_\omega = \rho^\omega(M_\omega)$, $M_\omega \subset C_\omega(M, \rho(M))$ follows. \square

Lemma 4.5. Assume that M is an infinite factor. Let $\rho \in \text{Cnd}(M)$. Then one has $\bar{\rho} \in \text{Cnd}(M)$.

Proof. First we show $\rho\bar{\rho} \in \text{Cnd}(M)$. By Lemma 4.3 and 4.4, $C(M, \rho\bar{\rho}(M)) = C(M, \rho(M)) = M_\omega$. This implies that for any $(x^\nu)_\nu \in \mathcal{C}_\omega(M, M)$, there exists $(y^\nu)_\nu \in \ell^\infty(\mathbf{N}, M)$ such that $x^\nu - \rho\bar{\rho}(y^\nu) \rightarrow 0$ strongly* as $\nu \rightarrow \omega$. Take an isometry $v \in (\text{id}, \rho\bar{\rho})$. Then $v^*x^\nu v - v^*\rho\bar{\rho}(y^\nu)v \rightarrow 0$ as $\nu \rightarrow \omega$. Here $v^*x^\nu v - x^\nu \rightarrow 0$ as $\nu \rightarrow \omega$ and $v^*\rho\bar{\rho}(y^\nu)v = y^\nu$. This yields that $x^\nu - y^\nu \rightarrow 0$, and $\rho\bar{\rho}(x^\nu) - x^\nu \rightarrow 0$ strongly* as $\nu \rightarrow \omega$. Hence $\rho\bar{\rho} \in \text{Cnd}(M)$.

Second we show that $\bar{\rho} \in \text{Cnd}(M)$. Since $\rho, \rho\bar{\rho} \in \text{Cnd}(M)$, $\rho(x^\nu) - \rho\bar{\rho}(x^\nu) \rightarrow 0$ strongly* as $\nu \rightarrow \omega$. Applying ϕ_ρ to $\rho(x^\nu) - \rho\bar{\rho}(x^\nu)$, we get the conclusion. \square

Theorem 4.6. Assume that M is an infinite factor. Then the subset $\text{Cnd}(M) \subset \text{End}(M)_0$ is closed under the composition, decomposition, direct sum and conjugation. Namely one has the following:

- (1) If $\rho, \sigma \in \text{Cnd}(M)$, then $\rho\sigma \in \text{Cnd}(M)$.
- (2) If $\rho \in \text{End}(M)_0$, $\sigma \in \text{Cnd}(M)$ and $\rho \prec \sigma$, then $\rho \in \text{Cnd}(M)$.
- (3) If $\rho, \sigma \in \text{Cnd}(M)$, then $\rho \oplus \sigma \in \text{Cnd}(M)$.
- (4) If $\rho \in \text{Cnd}(M)$, then $\bar{\rho} \in \text{Cnd}(M)$.

Proof. (1) It is trivial because $(\rho\sigma)^\omega = \rho^\omega\sigma^\omega = \text{id}$ on M_ω .

(2) Let $w \in (\rho, \sigma)$ be an isometry. Since $w \in (\rho^\omega, \sigma^\omega)$, we have for $x \in M_\omega$,

$$w\rho^\omega(x) = \sigma^\omega(x)w = xw = wx.$$

Hence $\rho^\omega(x) = w^*w\rho^\omega(x) = w^*wx = x$.

(3) Let $v, w \in M$ be isometries with $vv^* + ww^* = 1$. We set $\theta(x) := v\rho(x)v^* + w\sigma(x)w^*$ for $x \in M$. Then for $x \in M_\omega$, we have

$$\theta^\omega(x) = v\rho^\omega(x)v^* + w\sigma^\omega(x)w^* = vxv^* + wxw^* = (vv^* + ww^*)x = x.$$

Hence $\rho \oplus \sigma \in \text{Cnd}(M)$.

(4) It has been proved in the previous lemma. \square

Connes showed that any centrally trivial automorphism commutes with any approximately inner automorphism up to inner automorphism [5, Lemma 2.2.2]. We generalize this to the case of endomorphisms as follows.

Lemma 4.7. *Let M be a factor, $\rho \in \text{Cnd}(M)$ and $\theta \in \overline{\text{Int}}(M)$. Then θ and ρ commute up to inner automorphism, that is, $[\theta \circ \rho \circ \theta^{-1}] = [\rho]$ in $\text{Sect}(M)$.*

Proof. Let φ_0 be a faithful normal state on $\rho(M)$, and set $\varphi := \varphi_0 \circ E_\rho$. Let $\{\mathcal{V}_n\}_{n \in \mathbb{N}}$ be a fundamental system of the neighborhood of id in $\text{Aut}(M)$ such that for any $u \in U(M)$ with $\text{Ad } u \in \mathcal{V}_n$, we have

$$\|\rho(u) - u\|_{\varphi \circ \theta^{-1}} < 2^{-n}, \quad \|\rho(u) - u\|_{\varphi \circ \rho \circ \theta^{-1} \circ \phi_\rho} < 2^{-n}.$$

If β_n converges to θ in $\text{Aut}(M)$, then $\rho \circ \beta_n \circ \rho^{-1}$ converges to $\rho \circ \theta \circ \rho^{-1}$ in $\text{Aut}(\rho(M))$. So we can choose a monotone decreasing system of the neighborhood of θ denoted by \mathcal{W}_n such that $\mathcal{W}_n \mathcal{W}_n^{-1} \subset \mathcal{V}_n$ and for $\beta \in \mathcal{W}_n$, we have

$$\|\varphi \circ \beta^{-1} - \varphi \circ \theta^{-1}\| < 2^{-n}, \quad \|\varphi_0 \circ \rho \circ \beta^{-1} \circ \rho^{-1} - \varphi_0 \circ \rho \circ \theta^{-1} \circ \rho^{-1}\| < 2^{-n}.$$

Since $\varphi_0 = \varphi$ on $\rho(M)$, we get

$$\|\varphi \circ \rho \circ \beta^{-1} \circ \phi_\rho - \varphi \circ \rho \circ \theta^{-1} \circ \phi_\rho\| = \|\varphi \circ \rho \circ \beta^{-1} \circ \rho^{-1} \circ E_\rho - \varphi \circ \rho \circ \theta^{-1} \circ \rho^{-1} \circ E_\rho\| < 2^{-n}.$$

Let $\theta = \lim_{n \rightarrow \infty} \text{Ad } u_n$ with $\text{Ad } u_n \in \mathcal{W}_n$. Set $v_n := u_{n+1}u_n^*$. Then we see that $\text{Ad } v_n \in \mathcal{W}_{n+1}\mathcal{W}_{n+1}^{-1} \subset \mathcal{V}_n$. Hence $\|\rho(v_n) - v_n\|_{\varphi \circ \theta^{-1}} < 2^{-n}$ and $\|\rho(v_n) - v_n\|_{\varphi \circ \rho \circ \theta^{-1} \circ \phi_\rho} < 2^{-n}$.

Now we prove $u_n^*\rho(u_n)$ converges in $U(M)$. We have

$$\begin{aligned} \|\rho(u_{n+1}^*)u_{n+1} - \rho(u_n^*)u_n\|_\varphi^2 &= \|\rho(u_n^*v_n^*)v_nu_n - \rho(u_n^*)u_n\|_\varphi^2 = \|\rho(v_n^*)v_n - 1\|_{\varphi \circ \text{Ad } u_n^*}^2 \\ &\leq 2\|\varphi \circ \text{Ad } u_n^* - \varphi \circ \theta^{-1}\| + \|\rho(v_n^*)v_n - 1\|_{\varphi \circ \theta^{-1}}^2 \\ &\leq 2^{1-n} + \|v_n - \rho(v_n)\|_{\varphi \circ \theta^{-1}}^2 \leq 2^{1-n} + 4^{-n} \\ &\leq 2^{2-n}, \end{aligned}$$

and

$$\begin{aligned}
\|u_{n+1}^* \rho(u_{n+1}) - u_n^* \rho(u_n)\|_\varphi^2 &= \|u_n^* v_n^* \rho(v_n u_n) - u_n^* \rho(u_n)\|_\varphi^2 = \|v_n^* \rho(v_n) - 1\|_{\varphi \circ \text{Ad } \rho(u_n^*)}^2 \\
&\leq 2\|\varphi \circ \text{Ad } \rho(u_n^*) - \varphi \circ \rho \circ \theta^{-1} \circ \phi_\rho\| \\
&\quad + \|v_n^* \rho(v_n) - 1\|_{\varphi \circ \rho \circ \theta^{-1} \circ \phi_\rho}^2 \\
&\leq 2^{1-n} + 4^{-n} \\
&\leq 2^{2-n},
\end{aligned}$$

where we have used

$$\varphi \circ \text{Ad } \rho(u_n^*) = \varphi \circ \text{Ad } \rho(u_n^*) \circ E_\rho = \varphi \circ \rho \circ \text{Ad } u_n^* \circ \phi_\rho.$$

Hence the strong* limit $w := \lim_{n \rightarrow \infty} u_n^* \rho(u_n) \in U(M)$ exists, and we have $\theta^{-1} \circ \rho \circ \theta = \lim_{n \rightarrow \infty} \text{Ad } u_n^* \rho(u_n) \rho = \text{Ad } w \circ \rho$. \square

Let M be a McDuff factor, i.e., the von Neumann algebra M_ω is of type II_1 . When α is a centrally trivial automorphism of M , we know that α is outer conjugate to $\alpha \otimes \text{id}_{R_0}$, where R_0 denotes the hyperfinite type II_1 factor. This property also holds for centrally trivial endomorphisms.

Proposition 4.8. *Let M be an infinite McDuff factor and $\rho \in \text{Cnd}(M)$. Then there exists an isomorphism $\theta: M \otimes R_0 \rightarrow M$ such that $[\rho] = [\theta \circ (\rho \otimes \text{id}_{R_0}) \circ \theta^{-1}]$ in $\text{Sect}(M)$.*

Proof. If $[\rho] = [\text{id}]$, then the proposition is trivial. We assume that $[\rho] \neq [\text{id}]$. Take $\sigma \in \text{End}(M)$ such that $[\sigma] = [\text{id}] \oplus [\rho] \oplus [\rho]$. Then σ is centrally trivial by Theorem 4.6. Hence $C(M, \sigma(M)) = M_\omega$ holds by Lemma 4.4. Since M_ω is of type II_1 , $\sigma(M) \subset M$ is relatively McDuff in the sense of Bisch [1]. Thus $\sigma(M) \subset M$ is isomorphic to $(\sigma \otimes \text{id}_{R_0})(M \otimes R_0) \subset M \otimes R_0$. Hence we can find isomorphisms θ_1 and θ_2 from M onto $M \otimes R_0$ such that $\theta_1 \circ \sigma = (\sigma \otimes \text{id}_{R_0}) \circ \theta_2$. So,

$$[\theta_1 \theta_2^{-1}] \oplus 2[\theta_1 \circ \rho \circ \theta_1^{-1}] = [\theta_1 \sigma \theta_2^{-1}] = [\sigma \otimes \text{id}_{R_0}] = [\text{id} \otimes \text{id}_{R_0}] \oplus 2[\rho \otimes \text{id}_{R_0}]$$

holds. By comparing irreducible components, we have $[\theta_1 \theta_2^{-1}] = [\text{id} \otimes \text{id}_{R_0}]$, and $[\theta_1 \circ \rho \circ \theta_1^{-1}] = [\rho \otimes \text{id}_{R_0}]$. \square

Combining Lemma 3.11 with Proposition 4.8, we have the following result, where $R_{0,1}$ denotes the hyperfinite type II_∞ factor.

Proposition 4.9. *Let M be an infinite McDuff factor and $\rho \in \text{Cnd}(M)$. Then there exists an isomorphism $\theta: M \otimes R_{0,1} \rightarrow M$ such that $[\rho] = [\theta \circ (\rho \otimes \text{id}) \circ \theta^{-1}]$ in $\text{Sect}(M)$.*

4.2. Modular endomorphisms

Let M be a factor and $\rho \in \text{End}(M)_0$. We say that ρ is a *modular endomorphism* when the canonical extension $\tilde{\rho}$ is inner, that is, there exists a finite dimensional Hilbert space in \widetilde{M} which implements $\tilde{\rho}$ [17, Definition 3.1]. By $\text{End}(M)_m$ and $\text{Sect}(M)_m$, we denote the sets of modular endomorphisms on M and sectors of modular endomorphisms, respectively. We also denote by $\text{End}(M)_{m, \text{irr}}$ the set of

irreducible modular endomorphisms on M . Then $\text{End}(M)_m$ is closed under the composition, decomposition, direct sum and conjugation.

In type III case, Izumi characterizes modular endomorphisms in terms of $U(n)$ -valued cohomology classes $H^1(F^M, U(n))$ with respect to the flow of weights F^M [17, Theorem 3.3]. We complement his result for each type.

Lemma 4.10. *Let M be a factor. Then the following statements hold.*

- (1) *When M is of type II_1 , $\text{End}(M)_m = \text{Int}(M)$.*
- (2) *When M is of type II_∞ , $\text{End}(M)_{m, \text{irr}} = \text{Int}(M)$.*
- (3) *When M is of type III_λ ($0 < \lambda \leq 1$),*

$$\text{End}(M)_{m, \text{irr}} = \{\text{Ad } u \circ \sigma_t^\varphi \mid u \in U(M), \varphi \in W(M), t \in \mathbb{R}\}.$$

- (4) *When M is of type III_0 ,*

$$\text{End}(M)_m = \{\rho \in \text{End}(M)_0 \mid \delta_m([\rho]) \in H^1(F^M, U(n)), n \in \mathbb{N}\},$$

where δ_m is a bijection from $\text{Sect}(M)_m$ onto $\bigcup_{n \in \mathbb{N}} H^1(F^M, U(n))$ introduced in [17, p.10].

Proof. (3) and (4) is already proved [17, Theorem 3.3].

(1) Let τ be the normal tracial state on M . Then the natural isomorphism $\Pi_\tau: \widetilde{M} \rightarrow M \rtimes_{\sigma^\tau} \mathbb{R}$ maps $x \in M$ to $x \otimes 1$. Let $\rho \in \text{End}(M)_m$. Take $\{V_i\}_{i=1}^d \subset \widetilde{M}$ be an implementing orthonormal system of $\tilde{\rho}$. Then for $x \in M$, $(\rho(x) \otimes 1)V_i = \tilde{\rho}(x \otimes 1)V_i = V_i(x \otimes 1)$. This implies that $(\text{id}, \rho) \neq 0$. Since M is finite, (id, ρ) contains a unitary, and ρ is an inner automorphism.

(2) The above proof for $\rho \in \text{End}(M)_{m, \text{irr}}$ also works in this case. \square

Lemma 4.11. *Let M be a factor. Then $\text{End}(M)_m \subset \text{Cnd}(M)$.*

Proof. Let $\rho \in \text{End}(M)_m$ and $\psi \in W(M)$. We identify \widetilde{M} with $M \rtimes_{\sigma^\psi} \mathbb{R}$. Let $(x^\nu)_\nu$ be an ω -centralizing sequence in M . We will show that $\rho(x^\nu) - x^\nu \rightarrow 0$ strongly* as $\nu \rightarrow \omega$.

Note that $\pi_{\sigma^\psi}(x^\nu) - x^\nu \otimes 1 \rightarrow 0$ strongly as $\nu \rightarrow \omega$ in $M \otimes B(L^2(\mathbb{R}))$. It suffices to show $\|(\pi_{\sigma^\psi}(x^\nu) - x^\nu \otimes 1)(\xi \otimes f)\| \rightarrow 0$ for all $\xi \in L^2(M)$ and $f \in L^2(\mathbb{R})$ with compact support. It is checked as follows:

$$\begin{aligned} \|(\pi_{\sigma^\psi}(x^\nu) - x^\nu \otimes 1)(\xi \otimes f)\|^2 &= \int_{-\infty}^{\infty} |f(s)|^2 \|(\sigma_{-s}^\psi(x^\nu) - x^\nu)\xi\|^2 ds \\ &\rightarrow 0 \quad \text{as } \nu \rightarrow \omega, \end{aligned}$$

since $\|(\sigma_{-s}^\psi(x^\nu) - x^\nu)\xi\|$ uniformly converges to 0 on the compact support of f as $\nu \rightarrow \omega$ by [3, Proposition 2.3 (1)]. Similarly we can prove $\pi_{\sigma^\psi}((x^\nu)^*) - (x^\nu)^* \otimes 1 \rightarrow 0$. Hence $\pi_{\sigma^\psi}(x^\nu) - x^\nu \otimes 1 \rightarrow 0$ strongly* as $\nu \rightarrow \omega$. In particular, $(\pi_{\sigma^\psi}(x^\nu))_\nu$ is an ω -centralizing sequence in $M \otimes B(L^2(\mathbb{R}))$. Since $\tilde{\rho}$ is inner, $\tilde{\rho}$ acts trivially on ω -centralizing sequences of \widetilde{M} . Thus $\pi_\psi(\rho(x^\nu)) - \pi_\psi(x^\nu) = \tilde{\rho}(\pi_\psi(x^\nu)) - \pi_\psi(x^\nu)$ converges to 0 strongly* as $\nu \rightarrow \omega$. \square

Our main result in this section is the following.

Theorem 4.12. *Let M be a hyperfinite factor. Then one has*

$$\text{End}(M)_m = \text{Cnd}(M).$$

We present a proof separately for hyperfinite factors of type I, II_1 , II_∞ , III_1 , III_λ ($0 < \lambda < 1$) and III_0 . We denote by $\text{Cnd}(M)_{\text{irr}}$ the set of endomorphisms on M which are with finite index, centrally trivial and irreducible. By Theorem 4.6 and Lemma 4.11, it suffices to show that $\text{Cnd}(M)_{\text{irr}} \subset \text{End}(M)_m$.

4.3. Semifinite case

- *Proof of Theorem 4.12 for a hyperfinite type I factor.*

It is trivial because every endomorphism on M is inner. \square

- *Proof of Theorem 4.12 for the hyperfinite type II_∞ factor.*

Let M be the hyperfinite type II_∞ factor. We show that $\text{Cnd}(M)_{\text{irr}} \subset \text{Int}(M)$. Let $\rho \in \text{Cnd}(M)_{\text{irr}}$. We may assume ρ is of the form $\rho \otimes \text{id}_{R_{0,1}}$ by Proposition 4.9. If σ is a flip automorphism of $R_{0,1} \otimes R_{0,1}$, then σ is approximately inner. Hence $[\sigma \circ (\bar{\rho} \otimes \text{id}_{R_{0,1}}) \circ \sigma^{-1}] = [\bar{\rho} \otimes \text{id}_{R_{0,1}}]$ by Theorem 4.6 and Lemma 4.7. Then,

$$\begin{aligned} [\rho \otimes \bar{\rho}] &= [(\rho \otimes \text{id}_{R_{0,1}}) \circ \sigma \circ (\bar{\rho} \otimes \text{id}_{R_{0,1}}) \circ \sigma^{-1}] \\ &= [(\rho \otimes \text{id}_{R_{0,1}}) \circ (\bar{\rho} \otimes \text{id}_{R_{0,1}})] \\ &= [\rho \bar{\rho} \otimes \text{id}_{R_{0,1}}] \\ &\succ [\text{id}_{R_{0,1}} \otimes \text{id}_{R_{0,1}}]. \end{aligned}$$

Since ρ is irreducible, ρ is an inner automorphism. \square

- *Proof of Theorem 4.12 for the hyperfinite type II_1 factor.*

Let M be the hyperfinite type II_1 factor. We show that $\text{Cnd}(M) = \text{Int}(M)$. Let $\rho \in \text{Cnd}(M)$. On the type II_∞ factor $N = M \otimes B(\ell_2)$, we define the endomorphism $\sigma := \rho \otimes \text{id}$. We claim that σ is centrally trivial.

If not, there exists $x \in N_\omega$ such that $\sigma(x) \neq x$. Let $\{p_i\}_{i=1}^\infty$ be a partition of unity in $B(\ell_2)$ which consists of minimal projections. Then there exists $i \geq 1$ such that $\sigma^\omega(x)(1 \otimes p_i) \neq x(1 \otimes p_i)$. Since $x(1 \otimes p_i) = (1 \otimes p_i)x$ and $\sigma(1 \otimes p_i) = 1 \otimes p_i$, we have $\sigma^\omega((1 \otimes p_i)x(1 \otimes p_i)) \neq (1 \otimes p_i)x(1 \otimes p_i)$. Let $(x^\nu)_\nu$ be an ω -centralizing sequence for x . For each $\nu \in \mathbb{N}$, we take $y^\nu \in M$ such that $(1 \otimes p_i)x^\nu(1 \otimes p_i) = y^\nu \otimes p_i$. Then $(y^\nu)_\nu$ is an ω -centralizing sequence in M . Indeed, for $\varphi \in M_*$, $\psi = p_i \psi p_i \in B(\ell_2)_*$ and $z \in M$, we have

$$\begin{aligned} \psi(p_i)[\varphi, y^\nu](z) &= [\varphi \otimes \psi, y^\nu \otimes p_i](z \otimes 1) \\ &= [\varphi \otimes \psi, (1 \otimes p_i)x^\nu(1 \otimes p_i)](z \otimes 1) \\ &= [\varphi \otimes \psi, x^\nu](z \otimes 1), \end{aligned}$$

and $|\psi(p_i)|\|[\varphi, y^\nu]\| \leq \|[\varphi \otimes \psi, x^\nu]\|$. Hence $(y^\nu)_\nu$ is ω -centralizing.

Then $\sigma^\omega((1 \otimes p_i)x(1 \otimes p_i))$ is represented by $\sigma(y^\nu \otimes p_i) = \rho(y^\nu) \otimes p_i$ which is equivalent to $y^\nu \otimes p_i$ because ρ is centrally trivial. This is a contradiction with $\sigma^\omega((1 \otimes p_i)x(1 \otimes p_i)) \neq (1 \otimes p_i)x(1 \otimes p_i)$.

Hence $\rho \otimes \text{id}$ is centrally trivial, and inner by the above result on the hyperfinite type II_∞ factor. This implies $(\text{id}, \rho) \neq 0$. Since M is finite, (id, ρ) contains a unitary. Therefore ρ is an inner automorphism. \square

For type III cases, we prove Theorem 4.12 in the following subsections.

4.4. Type III_1 case

Central freeness for subfactors is introduced by Popa [29, Definition 3.1]. Following this notion, we prepare the notion of central non-triviality for subfactors.

Definition 4.13. Let $N \subset M$ be an inclusion of factors. Assume that there exists a faithful normal conditional expectation from M onto N . We say that $N \subset M$ is *centrally non-trivial* when the following condition is satisfied: for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset M_*$, there exists a partition of unity $(q_j)_{j=1}^t$ in N such that

$$\left\| \sum_{j=1}^t q_j \varphi q_j - \varphi \circ E_{N \vee (N' \cap M)}^M \right\| < \varepsilon \text{ for all } \varphi \in \mathcal{F},$$

where $E_{N \vee (N' \cap M)}^M$ is the unique faithful normal conditional expectation from M onto $N \vee (N' \cap M)$.

By definition, the central non-triviality is a condition that is independent of the choice of a particular conditional expectation from M onto N . The following lemma is proved in the same way as [29, Proposition 3.2].

Lemma 4.14. Let $N \subset M$ be an inclusion of factors. Let E_N^M be a faithful normal conditional expectation from M onto N . Fix a faithful normal state φ on N . Then the following statements are equivalent:

- (1) $N \subset M$ is centrally non-trivial.
- (2) For any $\delta, \varepsilon > 0$, any finite set $\mathcal{F} \subset N_*$, any finite family $(x_i)_{i=1}^m$ in the unit ball of M , there exists a partition of unity $(q_j)_{j=1}^t$ in N such that $t \leq (10^4 \varepsilon^{-1})^{6m \log \varepsilon^{-1}}$ and
 - (a) $\left\| \sum_{j=1}^t q_j \psi q_j - \psi \right\| < \delta$ for all $\psi \in \mathcal{F}$,
 - (b) $\left\| \sum_{j=1}^t q_j x_i q_j - E_{N \vee (N' \cap M)}^M(x_i) \right\|_{\varphi \circ E_N^M} < \varepsilon$ for all $1 \leq i \leq m$.

Lemma 4.15. Let M be an infinite factor. Let $\rho \in \text{Cnd}(M)_{\text{irr}}$ be non-inner. Set the endomorphism $\sigma := \text{id} \oplus \rho$. Then the inclusion $\sigma(M) \subset M$ is not centrally non-trivial.

Proof. Suppose that $\sigma(M) \subset M$ is centrally non-trivial. Then the locally trivial subfactor $N^{(\text{id}, \rho)} \subset M^{(\text{id}, \rho)}$ is centrally non-trivial because this inclusion is isomorphic to $\sigma(M) \subset M$. Since ρ is non-inner,

$$N^{(\text{id}, \rho)} \vee ((N^{(\text{id}, \rho)})' \cap M^{(\text{id}, \rho)}) = \{x \otimes e_{00} + \rho(y) \otimes e_{11} \mid x, y \in M\}.$$

This shows that the matrix unit $1 \otimes e_{01} \in M^{(\text{id}, \rho)}$ is orthogonal to $N^{(\text{id}, \rho)} \vee (N^{(\text{id}, \rho)})' \cap M^{(\text{id}, \rho)}$. Recall the embedding $\alpha^{(\text{id}, \rho)}: M \rightarrow M^{(\text{id}, \rho)}$ and the conditional expectation $E := E^{(1/2, 1/2)}$ as defined in §2.3. Let $\nu \in \mathbb{N}$ and $0 < \varepsilon < 1/\sqrt{2}$. Let $\{\mathcal{F}_\nu\}_{\nu=1}^\infty$ be an increasing finite subsets of M_* whose union is norm-dense. Then by central non-triviality, for each $\nu \in \mathbb{N}$, there exists a partition of unity $(q_j^\nu)_{j=1}^t$ in M , with $t \leq (10^4 \varepsilon^{-1})^{6 \log \varepsilon^{-1}}$, such that $\|[q_j^\nu, \chi]\| < 2/\nu$ for all $1 \leq j \leq t$, $\chi \in \mathcal{F}_\nu$ and

$$\varepsilon > \left\| \sum_{j=1}^t \alpha^{(\text{id}, \rho)}(q_j^\nu) (1 \otimes e_{01}) \alpha^{(\text{id}, \rho)}(q_j^\nu) \right\|_{\psi \circ (\alpha^{(\text{id}, \rho)})^{-1} \circ E} = \frac{1}{\sqrt{2}} \left\| \sum_{j=1}^t q_j^\nu \rho(q_j^\nu) \right\|_{\psi \circ \phi_\rho},$$

where ψ is a fixed faithful state in M_* . Since $(q_j^\nu)_\nu$ is centralizing, $\rho(q_j^\nu) - q_j^\nu \rightarrow 0$ strongly* as $\nu \rightarrow \omega$. Then we have

$$\frac{1}{\sqrt{2}} > \varepsilon \geq \lim_{\nu \rightarrow \omega} \frac{1}{\sqrt{2}} \left\| \sum_{j=1}^t q_j^\nu \rho(q_j^\nu) \right\|_{\psi \circ \phi_\rho} = \lim_{\nu \rightarrow \omega} \frac{1}{\sqrt{2}} \left\| \sum_{j=1}^t q_j^\nu \right\|_{\psi \circ \phi_\rho} = \frac{1}{\sqrt{2}}.$$

This is a contradiction. \square

Next we recall the following result [16, Theorem 3.5].

Theorem 4.16 (Izumi). *Let M be a type III_1 factor and $\sigma \in \text{End}(M)_0$. Then the following conditions are equivalent:*

- (1) $\sigma(M)' \cap M = \widetilde{\sigma}(\widetilde{M})' \cap \widetilde{M}$.
- (2) $[\sigma \bar{\sigma}]$ does not contain an outer modular automorphism.

Consider the following towers:

$$\begin{aligned} \mathbb{C} &= M' \cap M \subset \sigma(M)' \cap M \subset \sigma \bar{\sigma}(M)' \cap M \subset \cdots, \\ \mathbb{C} &= \widetilde{M}' \cap \widetilde{M} \subset \widetilde{\sigma}(\widetilde{M})' \cap \widetilde{M} \subset \widetilde{\sigma} \widetilde{\sigma}(\widetilde{M})' \cap \widetilde{M} \subset \cdots, \end{aligned}$$

We say that the graph change occurs at the stage n if the n -th algebras (\mathbb{C} is the 0th algebra) do not coincide. The following lemma is a direct consequence of the previous theorem.

Lemma 4.17. *Let M be a type III_1 factor and $\sigma \in \text{End}(M)_0$. Then the graph change of $\sigma(M) \subset M$ occurs at the stage $n \geq 0$ if and only if $[(\sigma \bar{\sigma})^n]$ contains an outer modular automorphism.*

The following theorem proved by Popa [29, Theorem 3.5 (ii), Corollary 4.4] is our key to characterize centrally trivial endomorphisms on the hyperfinite type III_1 factor. We note that Popa's theorem is proved for centrally free subfactors, but his proof still works for centrally non-trivial subfactors without any change.

Theorem 4.18 (Popa). *Let M be the hyperfinite type III_1 factor and $\sigma \in \text{End}(M)_0$. Let $\sigma(M) \subset M \subset M_1 \subset \cdots$ be the basic extension. Let $n \in \mathbb{N}$. Then the following statements are equivalent.*

- (1) *The n -th subfactor $M \subset M_n$ is centrally non-trivial.*
- (2) *The graph change does not occur at the stage less than or equal to n .*

(3) The sector $[(\sigma\bar{\sigma})^n]$ does not contain an outer modular automorphism.

Corollary 4.19. *Let M be the hyperfinite type III_1 factor and $\rho \in \text{Cnd}(M)_{\text{irr}}$. Then either the following (1) or (2) occurs:*

- (1) $[\rho] = [\sigma_t^\varphi]$ for some $t \in \mathbb{R}$.
- (2) $[\rho\bar{\rho}]$ contains an outer modular automorphism.

Proof. If ρ is inner, (1) follows. We consider the case that ρ is not inner. We set $\sigma := \text{id} \oplus \rho$. Then $\bar{\sigma}$ is also centrally trivial by Theorem 4.6. By Lemma 4.15, the inclusion $\bar{\sigma}(M) \subset M$ is not centrally non-trivial. Since the inclusion $\bar{\sigma}(M) \subset M$ is isomorphic to $M \subset M_1$, the graph change occurs at the 1st stage by Theorem 4.18. Hence $\sigma\bar{\sigma}$ contains an outer modular automorphism by Lemma 4.17. Using $[\sigma\bar{\sigma}] = [\text{id}] \oplus [\rho] \oplus [\bar{\rho}] \oplus [\rho\bar{\rho}]$, we see that one of $[\rho]$, $[\bar{\rho}]$ and $[\rho\bar{\rho}]$ contain an outer modular automorphism. \square

• *Proof of Theorem 4.12 for the hyperfinite type III_1 factor.*

It suffices to prove $\text{Cnd}(M)_{\text{irr}} \subset \text{End}(M)_{\text{m}}$ as before. Let $\rho \in \text{Cnd}(M)_{\text{irr}}$. We show that the condition (2) in Corollary 4.19 does not occur. Since ρ is irreducible, $\sigma_t^\varphi \prec \rho\bar{\rho}$ if and only if $[\rho] = [\sigma_t^\varphi \rho]$ in $\text{Sect}(M)$. We set $H := \{t \in \mathbb{R} \mid [\rho] = [\sigma_t^\varphi \rho]\}$. Then H is a subgroup of \mathbb{R} . Hence $t \in H$ implies $\sigma_{nt}^\varphi \prec \rho\bar{\rho}$ for all $n \in \mathbb{Z}$. Since M is of type III_1 and $\rho\bar{\rho}$ has finite index, this is possible only in the case $t = 0$, that is, the condition (1) in Corollary 4.19 holds. Hence $\rho \in \text{End}(M)_{\text{m}}$. \square

4.5. Type III_λ case ($0 < \lambda < 1$)

Lemma 4.20. *Let M be a McDuff factor of type III_λ , $0 < \lambda < 1$, and $\rho \in \text{Cnd}(M)$. Then for any $\alpha \in \text{Aut}(M)$, $[\alpha\rho] = [\rho\alpha]$ in $\text{Sect}(M)$.*

Proof. Since ρ is centrally trivial, there exists an isomorphism $\Psi: M \rightarrow M \otimes R_{0,1}$ such that $[\rho] = [\Psi^{-1} \circ (\rho \otimes \text{id}_{R_{0,1}}) \circ \Psi]$ in $\text{Sect}(M)$ by Proposition 4.9. Take $\theta_\mu \in \text{Aut}(R_{0,1})$ with $\text{mod}(\theta_\mu) = \mu > 0$. Define the automorphism $\alpha_\mu = \Psi^{-1} \circ (\text{id} \otimes \theta_\mu) \circ \Psi \in \text{Aut}(M)$. Then we have $[\rho\alpha_\mu] = [\Psi^{-1} \circ (\rho \otimes \theta_\mu) \circ \Psi] = [\alpha_\mu\rho]$. Note that $\text{mod}(\alpha_\mu) = \mu$ holds.

Let $\alpha \in \text{Aut}(M)$. Then there exists $\mu > 0$ such that $\text{mod}(\alpha) = \text{mod}(\alpha_\mu)$. We set $\beta := \alpha\alpha_\mu^{-1}$. Since $\text{mod}(\beta) = 1$, β is approximately inner [19, Theorem 1 (1)]. Hence we have $[\rho\beta] = [\beta\rho]$ by Lemma 4.7. Then,

$$[\rho\alpha] = [\rho\beta\alpha_\mu] = [\rho\beta][\alpha_\mu] = [\beta\rho][\alpha_\mu] = [\beta][\rho\alpha_\mu] = [\beta][\alpha_\mu\rho] = [\beta\alpha_\mu\rho] = [\alpha\rho].$$

\square

Let R be the hyperfinite type III_1 factor and φ a dominant weight. Then $M = R \rtimes_{\sigma_T^\varphi} \mathbb{Z}$ is a hyperfinite type III_λ factor with $T = -2\pi/\log \lambda$. Denote the implementing unitary of σ_T^φ by U . Let $\theta: \mathbb{T} = \mathbb{R}/[0, 1) \rightarrow \text{Aut}(M)$ be the dual action of σ_T^φ and $\psi = \hat{\varphi}$ the dual weight of φ . Then $\sigma_T^\psi = \text{Ad } U$ holds.

Lemma 4.21. *One has $R_\omega \subset M_\omega$ via the embedding $R \subset M$.*

Proof. Let $\chi \in R_*$ be a faithful state and $\hat{\chi}$ the dual state. Let $E_\theta: M \rightarrow R$ be the expectation obtained by averaging the \mathbb{T} -action θ . Then for any $x \in R$ and $y \in M$, we have $[\hat{\chi}, x](y) = [\chi, x](E_\theta(y))$. This shows that $\|[\hat{\chi}, x]\| \leq \|[\chi, x]\|$. Let $(x^\nu)_\nu$ be an ω -centralizing sequence in R . Then we have $[\hat{\chi}, x] \rightarrow 0$ as $\nu \rightarrow \omega$.

Since $\hat{\chi}$ is faithful, a set $\{\hat{\chi} \cdot (aU^n) \mid a \in R, n \in \mathbb{Z}\}$ spans a norm dense subset in M_* . To prove $(x^\nu)_\nu$ is an ω -centralizing sequence in M , it suffices to show that $[aU^n, x^\nu] \rightarrow 0$ strongly* for any $a \in R$ and $n \in \mathbb{Z}$. We have

$$\begin{aligned} [aU^n, x^\nu] &= [a, x^\nu]U^n + a[U^n, x^\nu] \\ &= [a, x^\nu]U^n + a(\sigma_T^\varphi(x^\nu) - x^\nu)U^n. \end{aligned}$$

Since $(x^\nu)_\nu$ is an ω -centralizing sequence in R and $\sigma_T^\varphi \in \text{Cnt}(R)$, the both terms converge to 0 strongly* as $\nu \rightarrow \omega$. \square

Lemma 4.22. *One has $\theta_p \notin \overline{\text{Int}}(M)$ for all $p \in \mathbb{T} \setminus \{0\}$. In particular, $\theta_p \notin \text{Cnt}(M)$ for all $p \in \mathbb{T} \setminus \{0\}$.*

Proof. It is well known that θ faithfully acts on the flow of weights of M . Hence θ_p is not in $\overline{\text{Int}}(M)$ for $p \neq 0$. \square

Now let $\rho \in \text{Cnd}(M)_{\text{irr}}$. For $p \in \mathbb{T}$, $[\theta_p \rho] = [\rho \theta_p]$ holds from Lemma 4.20. Hence there uniquely exists $\alpha_p \in \text{Int}(M)$ such that

$$\theta_p \rho = \alpha_p \rho \theta_p.$$

Let $\{\varphi_n\}_{n=1}^\infty$ be a norm dense sequence in the set of normal states on M . We prepare a function $\delta_\rho: \text{Aut}(M) \times \text{Aut}(M) \rightarrow [0, +\infty)$ defined by

$$\delta_\rho(\alpha, \beta) = \sum_{n=1}^\infty \frac{1}{2^n} \|\varphi_n \circ \phi_\rho \circ \alpha^{-1} - \varphi_n \circ \phi_\rho \circ \beta^{-1}\|.$$

Then δ_ρ defines a metric on $\text{Int}(M)$. Indeed, if $\alpha, \beta \in \text{Int}(M)$ satisfies $\delta_\rho(\alpha, \beta) = 0$, then $\alpha\rho = \beta\rho$. Take $w \in U(M)$ such that $\beta^{-1}\alpha = \text{Ad } w$. Then $\text{Ad } w \circ \rho = \rho$, and $w \in \rho(M)' \cap M = \mathbb{C}$. Hence $\alpha = \beta$. We call that topology the δ_ρ -topology.

Lemma 4.23. *The map $\alpha: \mathbb{T} \rightarrow \text{Int}(M)$ is continuous with respect to the δ_ρ -topology.*

Proof. Note that $\theta_p \circ \phi_\rho \circ \theta_p^{-1} = \phi_{\theta_p \rho \theta_p^{-1}} = \phi_{\alpha_p \rho} = \phi_\rho \circ \alpha_p^{-1}$. Then the statement is trivial because the map $\mathbb{T} \ni p \mapsto \chi \circ \phi_\rho \circ \alpha_p^{-1} = \chi \circ \theta_p \circ \phi_\rho \circ \theta_p^{-1} \in M_*$ is norm-continuous for any $\chi \in M_*$. \square

Lemma 4.24. *There exists $w \in U(M)$ such that*

$$\theta_p \circ (\text{Ad } w \circ \rho) = (\text{Ad } w \circ \rho) \circ \theta_p$$

holds for all $p \in \mathbb{T}$.

Proof. Let $U(M)/U(\mathbb{C})$ be the quotient Polish group. We have a bijective map $\overline{\text{Ad}}: U(M)/U(\mathbb{C}) \rightarrow \text{Int}(M)$ such that $\overline{\text{Ad}}([u]) = \text{Ad } u$ for $[u] \in U(M)/U(\mathbb{C})$, $u \in U(M)$. The map is continuous with respect to the δ_ρ -topology. By [34, Corollary A.10], the inverse map $\overline{\text{Ad}}^{-1}$ is Borel. Let $s: U(M)/U(\mathbb{C}) \rightarrow U(M)$

be a Borel cross section. We set $v_p := s(\overline{\text{Ad}}^{-1}(\alpha_p))^* \in U(M)$. Then the map $v: \mathbb{T} \rightarrow U(M)$ is Borel and satisfies

$$\theta_p \rho = \text{Ad } v_p^* \circ \rho \theta_p \quad \text{for all } p \in \mathbb{T}.$$

Now we set $\mu_{p,q} := v_p \theta_p(v_q) v_{p+q}^*$ for $p, q \in \mathbb{T}$. Then for any $x \in M$,

$$\begin{aligned} \mu_{p,q} \rho(x) \mu_{p,q}^* &= v_p \theta_p(v_q) v_{p+q}^* \rho(x) v_{p+q} \theta_p(v_q^*) v_p^* \\ &= v_p \theta_p(v_q) \theta_{p+q}(\rho(\theta_{-p-q}(x))) \theta_p(v_q^*) v_p^* \\ &= v_p \theta_p(v_q \theta_q(\rho(\theta_{-p-q}(x)))) v_q^* v_p^* \\ &= v_p \theta_p(\rho(\theta_{-p}(x))) v_p^* = \rho(x). \end{aligned}$$

Hence $\mu_{p,q} \in \rho(M)' \cap M = \mathbb{C}$. It is easy to see that μ satisfies

$$\mu_{p,q} \mu_{p+q,r} = \mu_{q,r} \mu_{p,q+r}, \quad \mu_{p,0} = 1 = \mu_{0,p}.$$

Hence $\mu: \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{C}$ is a 2-cocycle. By triviality of $H^2(\mathbb{T}, U(\mathbb{C}))$ (see [26, Proposition 2.1]), there exists a Borel map $\lambda: \mathbb{T} \rightarrow U(\mathbb{C})$ such that

$$\mu_{p,q} = \lambda_p \lambda_q \lambda_{p+q}^*$$

holds for $(p, q) \in \mathbb{T} \times \mathbb{T}$ in the outside of a null set $N \subset \mathbb{T} \times \mathbb{T}$ with respect to the Haar measure. We set $\bar{v}_p := \lambda_p^* v_p$ for $p \in \mathbb{T}$. Then the map $\bar{v}: \mathbb{T} \rightarrow U(M)$ is Borel and satisfies $\theta_p \rho = \text{Ad } \bar{v}_p^* \circ \rho \theta_p$ for every $p \in \mathbb{T}$ and

$$\bar{v}_p \theta_p(\bar{v}_p) = \bar{v}_{p+q} \quad \text{for all } (p, q) \in \mathbb{T} \times \mathbb{T} \setminus N.$$

Hence \bar{v} satisfies a 1-cocycle relation almost everywhere, and it coincides with a 1-cocycle \bar{v}' almost everywhere (see [7, Remark III.1.9] and [27]). Since θ is a minimal action of the compact group \mathbb{T} and $M^\theta = R$ is purely infinite, θ is stable (see [17, Proposition 5.2]). Hence there exists $w \in U(M)$ such that $\bar{v}'_p = w^* \theta_p(w)$ holds for almost every $p \in \mathbb{T}$, and

$$\theta_p \circ (\text{Ad } w \circ \rho) = (\text{Ad } w \circ \rho) \circ \theta_p$$

holds for almost every $p \in \mathbb{T}$. By continuity of θ , it holds for every $p \in \mathbb{T}$. \square

• *Proof of Theorem 4.12 for the hyperfinite type III_λ factor ($0 < \lambda < 1$).*

It suffices to prove $\text{Cnd}(M)_{\text{irr}} \subset \text{End}(M)_{\text{m}}$ as before. Let $\rho \in \text{Cnd}(M)_{\text{irr}}$. By Lemma 4.24, we may and do assume that ρ commutes with θ :

$$\theta_p \rho = \rho \theta_p.$$

Hence $\sigma := \rho|_R \in \text{End}(R)$ has the meaning. Note that U and $\rho(U)$ are normalizing R in M . Indeed, the commutativity of θ with ρ yields $\rho(U) R \rho(U^*) \subset M^\theta = R$.

We consider the relative commutant $\sigma(R)' \cap R$. This is finite dimensional by the Pimsner-Popa inequality for $\rho \circ \phi_\rho$. Since ρ is irreducible,

$$(\sigma(R)' \cap R)^{\text{Ad } \rho(U)} = \rho(M)' \cap R = \mathbb{C}.$$

This means that the finite dimensional von Neumann algebra $\sigma(R)' \cap R$ admits the ergodic \mathbb{Z} -action $\text{Ad } \rho(U)$. Hence $\sigma(R)' \cap R$ is abelian. Let $\dim(\sigma(R)' \cap R) = n$ and p_1, \dots, p_n be the minimal projections of $\sigma(R)' \cap R$ such that

$$\sigma(R)' \cap R = \mathbb{C}p_1 + \dots + \mathbb{C}p_n.$$

By Lemma 4.21, $R_\omega \subset M_\omega$ holds. Hence $\sigma \in \text{End}(R)_0$ is centrally trivial. Applying Theorem 4.12 for the hyperfinite type III₁ factor R , we see that σ is a modular endomorphism, that is, there exists $t_1, \dots, t_n \in \mathbb{R}$ such that

$$[\sigma] = \bigoplus_{i=1}^n [\sigma_{t_i}^\varphi] \in \text{Sect}(R).$$

Take an isometry $w_i \in (\sigma_{t_i}^\varphi, \sigma)$ with $w_i w_i^* = p_i$ for each i . Then we have

$$\sigma(x) = \sum_{i=1}^n w_i \sigma_{t_i}^\varphi(x) w_i^* \quad \text{for all } x \in R.$$

Set $\mu = \sum_{i=1}^n \sigma_T^\varphi(w_i) w_i^* \in U(R)$. Then we have

$$\sigma_T^\varphi \sigma \sigma_{-T}^\varphi = \text{Ad } \mu \circ \sigma.$$

Since $\text{Ad } U|_R = \sigma_T^\varphi$, we have

$$\text{Ad } U \rho(U^*) \circ \sigma = \text{Ad } \mu \circ \sigma,$$

where we note that $U \rho(U^*) \in M^\theta = R$, and $\mu^* U \rho(U^*) \in \sigma(R)' \cap R$. Hence there exists $\chi_1, \dots, \chi_n \in \mathbb{T}$ such that

$$U \rho(U^*) = \mu \cdot \left(\sum_{i=1}^n \chi_i p_i \right) = \sum_{i=1}^n \chi_i \sigma_T^\varphi(w_i) w_i^* = \sum_{i=1}^n \chi_i U w_i U^* w_i^*,$$

and we have

$$\rho(U) = \sum_{i=1}^n \overline{\chi_i} w_i U w_i^*.$$

Taking $s_i \in \mathbb{T}$ with $\theta_{s_i}(U) = \overline{\chi_i} U$, we have

$$\rho(U) = \sum_{i=1}^n w_i \theta_{s_i}(U) w_i^*.$$

Therefore ρ has the following decomposition:

$$\rho(x) = \sum_{i=1}^n w_i \sigma_{t_i}^\psi(\theta_{s_i}(x)) w_i^* \quad \text{for all } x \in M.$$

Since ρ is irreducible, we have $n = 1$ and $[\rho] = [\sigma_{t_1}^\psi \theta_{s_1}]$. This shows θ_{s_1} must be centrally trivial. By Lemma 4.22, $s_1 = 0$, and we have $[\rho] = [\sigma_{t_1}^\psi]$. Hence ρ is a modular automorphism. \square

4.6. Type III₀ case

4.6.1. *Reduction of the problem to the study of type II inclusions.* Let $M = N \rtimes_{\theta} \mathbb{Z}$ be the discrete decomposition of a type III₀ factor M with the implementing unitary U . Let τ be a trace on N and $\varphi = \hat{\tau}$ be the dual weight on M . Then it is known that the weight φ is lacunary. Let $\hat{\theta}$ be the dual action of the torus \mathbb{T} on M . By [12, Theorem 3.1, Corollary 4.6], $\hat{\theta}_p$ is approximately inner for $p \in \mathbb{T}$.

Lemma 4.25. *Let $M = N \rtimes_{\theta} \mathbb{Z}$ be the discrete decomposition as before. Let $\rho \in \text{Cnd}(M)_{\text{irr}}$. Then there exists a unitary $w \in M$ such that $\text{Ad } w\rho$ and $\hat{\theta}$ commute and $\text{Ad } w\rho(U) = U$.*

Proof. Since $\hat{\theta}_p$ is approximately inner, ρ and $\hat{\theta}_p$ commute in $\text{Sect}(M)$ by Lemma 4.7. Through the \mathbb{T} -valued 2-cohomology vanishing as in type III _{λ} case, we can take a $\hat{\theta}$ -cocycle v such that

$$\hat{\theta}_p \rho \hat{\theta}_p^{-1} = \text{Ad } v_p^* \rho \quad \text{for all } p \in \mathbb{T}.$$

We show that v is a $\hat{\theta}$ -coboundary by using the 2-by-2 matrix argument. Set $P := M_2(\mathbb{C}) \otimes M$, the \mathbb{T} -action $\alpha := \text{id} \otimes \hat{\theta}$ and the α -cocycle $w := e_{11} \otimes 1 + e_{22} \otimes v$. Let $\beta = \text{Ad } v\alpha$ be the perturbed action. We will show $p := e_{11} \otimes 1$ and $q := e_{22} \otimes 1$ are equivalent in P^β .

First we show that p and q are properly infinite projections in P^β . Since $pP^\beta p = \mathbb{C}e_{11} \otimes M^{\hat{\theta}}$, p is properly infinite. For q , we have $qP^\beta q = \mathbb{C}e_{22} \otimes M^{\text{Ad } v\hat{\theta}}$. Since $M^{\text{Ad } v\hat{\theta}} \supset \rho(M^{\hat{\theta}})$, q is properly infinite.

Second we show that the central supports of p and q are equal to 1. We set a unitary $V := e_{11} \otimes U + e_{22} \otimes \rho(U)$. The equality $\beta_p(V) = e^{-ip}V$ shows that β is a dual action. Hence P is naturally isomorphic to $P^\beta \rtimes_{\text{Ad } V} \mathbb{Z}$. We regard $P = P^\beta \rtimes_{\text{Ad } V} \mathbb{Z}$. On one hand, $P \rtimes_{\beta} \mathbb{T}$ is isomorphic to $P^\beta \otimes B(\ell^2(\mathbb{Z}))$ by Takesaki duality. Since $p, q \in P^\beta$, $\pi_\beta(p)$ and $\pi_\beta(q)$ are mapped to $p \otimes 1$ to $q \otimes 1$. Hence it suffices to show that the central supports of $\pi_\beta(p)$ and $\pi_\beta(q)$ are equal to 1 in $P \rtimes_{\beta} \mathbb{T}$. On the other hand, $P \rtimes_{\beta} \mathbb{T}$ is naturally isomorphic to $P \rtimes_{\alpha} \mathbb{T} = M_2(\mathbb{C}) \otimes (M \rtimes_{\hat{\theta}} \mathbb{T})$. Since $p, q \in P^\alpha \cap P^\beta$, $\pi_\beta(p)$ and $\pi_\beta(q)$ are mapped to $\pi_\alpha(p) = p \otimes 1 = e_{11} \otimes 1 \otimes 1$ and $\pi_\alpha(q) = q \otimes 1 = e_{22} \otimes 1 \otimes 1$. Hence the central supports of $\pi_\beta(p)$ and $\pi_\beta(q)$ in $P \rtimes_{\beta} \mathbb{T}$ are equal to 1.

Therefore p and q are equivalent in P^β , and v is a $\hat{\theta}$ -coboundary. Take a unitary $w \in M$ such that $v_p = w^* \theta_p(w)$. Then $\hat{\theta}_p \circ (\text{Ad } w\rho) = (\text{Ad } w\rho) \circ \hat{\theta}_p$. Hence we may assume that ρ and $\hat{\theta}_p$ commute. Then $\rho(U)U^*$ is contained in $M^{\hat{\theta}} = N$. Since θ is stable [7, Theorem III.5.1 (i)], there exists $w_1 \in U(N)$ such that $\rho(U)U^* = w_1^* \theta(w_1)$. Then $\text{Ad } w_1 \rho$ satisfies the desired properties. \square

Lemma 4.26. *Let ρ be an irreducible endomorphism with finite index on M . Assume that ϕ_ρ and $\hat{\theta}$ commute and $\rho(U) = U$. If an endomorphism $\rho|_N \theta^n$ on N is inner for some $n \in \mathbb{Z}$, then ρ is a modular endomorphism.*

Proof. The proof is similar to that of [12, Theorem 5.2]. We may and do assume that $\rho|_N$ is inner by perturbing ρ to $\rho \text{Ad } U^n$. Let $\mathcal{H} \subset N$ be a Hilbert space

such that $\rho|_N = \rho_{\mathcal{H}}$. Since ρ has finite index, \mathcal{H} is finite dimensional. Put $d := \dim(\mathcal{H})$. Let $(V_i)_{i=1}^d$ be an orthonormal base of \mathcal{H} . Note that $\varphi\rho = d\varphi$ holds. Indeed for $x \in M_+$, we have

$$\begin{aligned}\varphi(\rho(x)) &= \tau\left(\int_{\mathbb{T}} \hat{\theta}_p(\rho(x))\right) = \tau\left(\rho\left(\int_{\mathbb{T}} \hat{\theta}_p(x)\right)\right) \\ &= \sum_{i=1}^d \tau\left(V_i\left(\int_{\mathbb{T}} \hat{\theta}_p(x)\right)V_i^*\right) = \sum_{i=1}^d \tau\left(V_i^*V_i\left(\int_{\mathbb{T}} \hat{\theta}_p(x)\right)\right) \\ &= d\tau\left(\int_{\mathbb{T}} \hat{\theta}_p(x)\right) = d\varphi(x).\end{aligned}$$

Next we will show $\tau \circ \phi_\rho = d^{-1}\tau$ on N . Let h be the positive operator affiliated with N such that $\tau \circ \phi_\rho = \tau_h$. Since $\phi_\rho\rho|_N = \text{id}_N$, h is indeed contained in $\rho(N)' \cap N$. Moreover we have

$$\begin{aligned}\theta(h^{it}) &= \theta([D\tau \circ \phi_\rho : D\tau]_t) = [D\tau \circ \phi_\rho \circ \theta^{-1} : D\tau\theta^{-1}]_t = [D\tau\theta^{-1} \circ \phi_\rho : D\tau\theta^{-1}]_t \\ &= [D\tau\theta^{-1} \circ \phi_\rho : D\tau \circ \phi_\rho]_t [D\tau \circ \phi_\rho : D\tau]_t [D\tau : D\tau\theta^{-1}]_t \\ &= \rho([D\tau\theta^{-1} : D\tau]_t) [D\tau \circ \phi_\rho : D\tau]_t [D\tau : D\tau\theta^{-1}]_t \\ &= [D\tau\theta^{-1} : D\tau]_t [D\tau \circ \phi_\rho : D\tau]_t [D\tau : D\tau\theta^{-1}]_t \\ &= [D\tau \circ \phi_\rho : D\tau]_t [D\tau\theta^{-1} : D\tau]_t [D\tau : D\tau\theta^{-1}]_t \\ &= [D\tau\phi_\rho : D\tau]_t = h^{it}.\end{aligned}$$

Hence h is affiliated with $(\rho(N)' \cap N)^\theta = \rho(M)' \cap N = \mathbb{C}$, and h is a positive scalar. Using $\tau\rho = d\tau$ and $\tau \circ \phi_\rho = h\tau$, we have $h = d^{-1}$. Since $\hat{\theta}$ and ϕ_ρ commute, we have $\varphi \circ \phi_\rho = d^{-1}\varphi$.

Define $\phi_1 = d^{-1} \sum_{i=1}^d V_i^* x V_i$ for $x \in N$. Then ϕ_1 is a left inverse for $\rho|_N$ and satisfies $\tau \circ \phi_1 = d^{-1}\tau$. The equality $\tau \circ \phi_\rho|_N = \tau \circ \phi_1$ implies that $\phi_\rho|_N = \phi_1$.

Now we show that $d(\rho) = d$. From the Pimsner-Popa inequality, the map $\rho \circ \phi_\rho|_N - d(\rho)^{-2} \text{id}_N$ is a completely positive on N . Set a projection $p := \sum_{i,j=1}^d d^{-1} V_i V_i^* V_j^* V_j \in N$. Then $\rho(\phi_\rho(p)) = \rho_{\mathcal{H}}(\phi_1(p)) = d^{-2}$. Hence $d(\rho) \geq d$.

Note that $\rho(\phi_\rho(U)) = U$. Regarding $M = N \rtimes_{\theta} \mathbb{Z} \subset N \otimes B(\ell^2(\mathbb{Z}))$, we can see that $(\rho_{\mathcal{H}} \circ \phi_1 \otimes \text{id})|_M$ is a conditional expectation from M onto $\rho(M)$ whose index is equal to d^2 . By the minimality of E_ρ , $d^2 \geq \text{Ind}(E_\rho) = d(\rho)^2$. Thus $d(\rho) = d$.

The equality $\rho(U) = U$ is equivalent to $\mathcal{H}^*\theta(\mathcal{H}) \subset Z(N)$. We set $c_{i,j} := V_i^*\theta(V_j)$ for $1 \leq i, j \leq d$. Then $c := (c_{i,j})_{i,j}$ is unitary in $M_d(\mathbb{C}) \otimes Z(N)$.

Consider the canonical core $\widetilde{M} = M \rtimes_{\varphi} \mathbb{R} \subset M \otimes B(L^2(\mathbb{R}))$ and the canonical extension $\widetilde{\rho}$. Note that $Z(N) \otimes L(\mathbb{R}) \subset \widetilde{M}$. Let $\beta = \pi_{\sigma^\varphi}(U) \in \text{Aut}(\widetilde{M})$. Then it is known that the covariant system $\{Z(N) \otimes L(\mathbb{R}), \beta\}$ has a fundamental domain, that is, there exists a projection $e \in Z(N) \otimes L(\mathbb{R})$ such that $\sum_{n \in \mathbb{Z}} \beta^n(e) = 1$. In particular, $\{M_d(\mathbb{C}) \otimes Z(N) \otimes L(\mathbb{R}), \text{id} \otimes \beta\}$ is stable. Hence there exists a unitary $\nu \in M_d(\mathbb{C}) \otimes Z(N) \otimes L(\mathbb{R})$ such that $c \otimes 1 = \nu(\text{id} \otimes \beta)(\nu^*)$. Set $W_i := \sum_{j=1}^d \pi_{\sigma^\varphi}(V_j) \nu_{ji}$. It is easy to see that a family $(W_i)_{i=1}^d$ span a Hilbert space in \widetilde{M} .

We show that $\tilde{\rho}$ is implemented by $(W_i)_{i=1}^d$. Take $x \in N$. Since $\nu_{ij} \in Z(N) \otimes L(\mathbb{R})$ for all i, j and $Z(N) \otimes L(\mathbb{R}) \subset \pi_{\sigma^\varphi}(N)'$, we have

$$W_i \pi_{\sigma^\varphi}(x) = \pi_{\sigma^\varphi}(\rho(x)) W_i = \tilde{\rho}(\pi_{\sigma^\varphi}(x)) W_i.$$

Since $\varphi \circ \phi_\rho = d^{-1}\varphi = d(\rho)^{-1}\varphi$, we have $\tilde{\rho}(\lambda^\varphi(t)) = \lambda^\varphi(t)$. Then it is trivial that

$$W_i \lambda^\varphi(t) = \lambda^\varphi(t) W_i = \tilde{\rho}(\lambda^\varphi(t)) W_i \quad \text{for all } t \in \mathbb{R}.$$

Finally we have

$$\begin{aligned} \sum_{i=1}^d W_i \pi_{\sigma^\varphi}(U) W_i^* &= \sum_{i,j,k=1}^d \pi_{\sigma^\varphi}(V_j) \nu_{ji} \pi_{\sigma^\varphi}(U) \nu_{ki}^* \pi_{\sigma^\varphi}(V_k^*) \\ &= \sum_{i,j,k=1}^d \pi_{\sigma^\varphi}(V_j) \nu_{ji} \beta(\nu_{ki}^*) \pi_{\sigma^\varphi}(U) \pi_{\sigma^\varphi}(V_k^*) \\ &= \sum_{j,k=1}^d \pi_{\sigma^\varphi}(V_j) (\nu(\text{id} \otimes \beta)(\nu^*))_{jk} \pi_{\sigma^\varphi}(\theta(V_k^*) U) \\ &= \sum_{j,k=1}^d \pi_{\sigma^\varphi}(V_j) (c \otimes 1)_{jk} \pi_{\sigma^\varphi}(\theta(V_k^*) U) \\ &= \sum_{j,k}^d \pi_{\sigma^\varphi}(V_j) (c_{jk} \otimes 1) \pi_{\sigma^\varphi}(\theta(V_k^*) U) \\ &= \sum_{j,k}^d \pi_{\sigma^\varphi}(V_j c_{jk} \theta(V_k^*) U) = \sum_{j,k}^d \pi_{\sigma^\varphi}(V_j V_j^* \theta(V_k V_k^*) U) \\ &= \pi_{\sigma^\varphi}(U) = \tilde{\rho}(\pi_{\sigma^\varphi}(U)). \end{aligned}$$

Since the core \widetilde{M} is generated by $\pi_{\sigma^\varphi}(N)$, $\pi_{\sigma^\varphi}(U)$ and $\lambda^\varphi(t) = 1 \otimes \lambda(t)$, $t \in \mathbb{R}$, we see that $\tilde{\rho}$ is implemented by $(W_i)_{i=1}^d$. \square

4.6.2. Central non-triviality of hyperfinite type II inclusions.

Lemma 4.27. *Let $N \subset M$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation E_N^M . Assume that N is hyperfinite and of type II. Then the inclusion is centrally non-trivial in the following sense: For any $\delta, \varepsilon > 0$, any faithful state φ on N , any finite subset $\mathcal{F} \subset N_*$ and any finite family $(x_i)_{i=1}^m$ in the unit ball of M with $E_{N \vee (N' \cap M)}^M(x_i) = 0$, there exists a partition of unity $(q_j)_{j=1}^t$ in N such that*

(1) *the partition number t does not depend on δ and \mathcal{F} .*

(2) $\left\| \sum_{j=1}^t q_j x_i q_j \right\|_{\varphi \circ E_N^M} < \varepsilon$ *for all $1 \leq i \leq m$,*

(3) $\left\| \sum_{j=1}^t q_j \chi q_j - \chi \right\| < \delta$ *for all $\chi \in \mathcal{F}$.*

Proof. This can be similarly proved as in the case of subfactors [29]. We may and do assume that N is of type II_1 (see [29, Proposition 3.4 (i)]). Let τ be a faithful tracial state on N . Let \mathcal{C}_N be the center of N . Since N is hyperfinite, we can regard $N = \mathcal{C}_N \otimes R_0$, where R_0 denotes the hyperfinite type II_1 factor. Let R_n be a type I_{2^n} subfactor of R_0 such that $\{R_n\}_{n=1}^\infty$ is an increasing sequence and $\cup_{n \geq 1} R_n$ is weakly dense in R_0 .

We set $\mathcal{N}_n := R'_n \cap N$ and $\mathcal{M}_n := R'_n \cap M$. The inclusion $N \subset M$ is isomorphic to $\mathcal{N}_n \otimes R_n \subset \mathcal{M}_n \otimes R_n$, and the state ψ is $\psi|_{\mathcal{M}_n} \otimes \tau_{R_n}$. Using this identification, we have

$$N \vee (N' \cap M) = (\mathcal{N}_n \vee (\mathcal{N}'_n \cap \mathcal{M}_n)) \otimes R_n = \mathcal{N}_n \vee (\mathcal{N}'_n \cap M).$$

In particular, this implies $E_{N \vee (N' \cap M)}^M = E_{\mathcal{N}_n \vee (\mathcal{N}'_n \cap M)}^M$.

Let $\mathcal{F} := \{\chi_i\}_{i=1}^k \subset N_*$ be a finite set. Let $\delta > 0$. Since the union of an increasing sequence $\{\mathcal{C}_N \otimes R_n\}_{n=1}^\infty$ is strongly dense in N , the set $\{\tau a \mid a \in \mathcal{C}_N \otimes R_n, n \geq 1\}$ is dense in N_* . Hence there exists $p_0 \in \mathbb{N}$ and $a_i \in \mathcal{C}_N \otimes R_{p_0}$, $1 \leq i \leq k$ such that $\|\chi_i - \tau a_i\| < \delta/2$ for $1 \leq i \leq k$.

Let $\varepsilon > 0$. By spectral analysis, there exists a positive invertible h in N such that $\|\varphi - \tau h\| < \varepsilon^2/4$ and $\tau(h) = 1$. By taking enough large $p_1 \in \mathbb{N}$, there exists a positive invertible $h_1 \in \mathcal{C}_N \otimes R_{p_1}$ such that $\|\tau h - \tau h_1\| < \varepsilon^2/4$ and $\tau(h_1) = 1$. Then $\|\varphi - \tau h_1\| < \varepsilon^2/2$. We set $\psi := \tau h_1 \in N_*$. Putting $p := \max(p_0, p_1)$, we have $h, a_i \in \mathcal{C}_N \otimes R_p$.

Next let $(x_i)_{i=1}^m$ be given as in the statement. Then $(x_i)_{i=1}^m$ are orthogonal to $\mathcal{N}_p \vee (\mathcal{N}'_p \cap M)$. Applying Popa's local quantization to $\mathcal{N}_p \subset M_{\psi \circ E_N^M}$ [29, Theorem A.1.2], we have a partition of unity $(q_j)_{j=1}^t$ in \mathcal{N}_p such that

$$\left\| \sum_{j=1}^t q_j x_i q_j \right\|_{\psi \circ E_N^M} < \frac{\varepsilon}{\sqrt{2}} \quad \text{for all } 1 \leq i \leq m,$$

where $t \leq (40\sqrt[4]{2}\varepsilon^{-1/2})^{(m \log(2\varepsilon^{-2})/\log \frac{4}{3})}$. We check that $(q_j)_{j=1}^t$ has the desired property. Since $\left\| \sum_{j=1}^t q_j x_i q_j \right\| \leq \|x_i\| \leq 1$, we have

$$\begin{aligned} \left\| \sum_{j=1}^t q_j x_i q_j \right\|_{\varphi \circ E_N^M}^2 &= \varphi \circ E_N^M \left(\left| \sum_{j=1}^t q_j x_i q_j \right|^2 \right) \\ &= (\varphi \circ E_N^M - \psi \circ E_N^M) \left(\left| \sum_{j=1}^t q_j x_i q_j \right|^2 \right) + \left\| \sum_{j=1}^t q_j x_i q_j \right\|_{\psi \circ E_N^M}^2 \\ &\leq \|\varphi \circ E_N^M - \psi \circ E_N^M\| + \varepsilon^2/2 \\ &= \|\varphi - \psi\| + \varepsilon^2/2 < \varepsilon^2/2 + \varepsilon^2/2 = \varepsilon^2. \end{aligned}$$

Hence we have

$$\left\| \sum_{j=1}^t q_j x_i q_j \right\|_{\varphi \circ E_N^M} < \varepsilon.$$

Moreover, since $q_j \in \mathcal{N}_p$ and $a_i \in \mathcal{C}_N \otimes R_p$ commute, we have

$$\|[\chi_i, q_j]\| = \|[\chi_i - \tau a_i, q_j]\| \leq 2\|\chi_i - \tau a_i\| < \delta.$$

□

The previous lemma immediately implies the following lemma.

Lemma 4.28. *Let $N \subset M$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation E_N^M . Assume that N is hyperfinite and of type II. Then for any $\varepsilon > 0$, any faithful state φ on N and any finite family $(x_i)_{i=1}^m$ in the unit ball of M which satisfies $E_{N \vee (N' \cap M)}^M(x_i) = 0$ and $x_i \neq 0$, there exist $t \in \mathbb{N}$ and a partition of unity $(q_j)_{j=1}^t$ in N_ω such that*

$$\left\| \sum_{j=1}^t q_j x_i q_j \right\|_{(\varphi \circ E_N^M)^\omega} < \varepsilon \|x_i\|_{\varphi \circ E_N^M} \quad \text{for all } 1 \leq i \leq m.$$

Proof. In the previous lemma, we take $(\varepsilon/2) \min\{\|x_i\|_{\varphi \circ E_N^M} \mid 1 \leq i \leq m\}$ for ε . Note that we can take the partition number t which does not depend on δ and \mathcal{F} . Let $\{\mathcal{F}_\nu\}_{\nu=1}^\infty$ be an increasing sequence of finite sets in N_* whose union is dense in N_* . Letting $\delta = 1/\nu$ and $\mathcal{F} = \mathcal{F}_\nu$ for $\nu \in \mathbb{N}$ in the previous lemma, we obtain a corresponding partition of unity $(q_j^\nu)_{j=1}^t$. Then it is trivial that the sequence $(q_j^\nu)_\nu$ is centralizing, and $q_j := \pi_\omega((q_j^\nu)_\nu)$, $1 \leq j \leq t$ is a desired partition of unity. □

4.6.3. Endomorphisms on type II von Neumann algebras. Let N be a hyperfinite type II von Neumann algebra with a centrally ergodic automorphism θ . We denote by \mathcal{C} the center of N . We consider a subset $\mathcal{E}_\theta \subset \text{End}(N)$ which consists of endomorphisms with left inverses commuting θ . For each $\rho \in \mathcal{E}_\theta$, we choose a left inverse ϕ_ρ on N such that $\phi_\rho \theta = \theta \phi_\rho$. Note that this equality implies $\rho \theta = \theta \rho$. For endomorphisms ρ, σ on N , we write $(\rho, \sigma)_N$ for (ρ, σ) to specify N .

Lemma 4.29. *Assume that $\rho \in \mathcal{E}_\theta$ satisfies $(\rho, \theta^n)_N = 0$ for any $n \in \mathbb{Z}$. Then for any $\varepsilon > 0$, $m \in \mathbb{N}$ and a faithful state $\psi \in N_*$, there exists $t \in \mathbb{N}$ and a partition of unity $(q_i)_{i=0}^t$ in N_ω such that*

- (1) $\sum_{|n| \leq m} |q_i \rho^\omega(\theta_\omega^n(q_i))|_{(\psi \circ \phi_\rho)^\omega} < \varepsilon |q_i|_{(\psi \circ \phi_\rho)^\omega}$ for all $1 \leq i \leq t$,
- (2) $|q_0|_{(\psi \circ \phi_\rho)^\omega} < \varepsilon$.

Proof. Set $\varphi := \psi \circ \phi_\rho \in N_*$ and $\mathcal{M} := M_{2m+2}(\mathbb{C}) \otimes M$. Let $\{e_{ij} \mid -m \leq i, j \leq m+1\}$ be a system of matrix units of $M_{2m+2}(\mathbb{C})$. We introduce a homomorphism $\pi: N \rightarrow \mathcal{M}$ defined by

$$\pi(x) = \sum_{i=-m}^{-1} e_{ii} \otimes \rho(\theta^i(x)) + e_{00} \otimes x + \sum_{i=1}^{m+1} e_{ii} \otimes \rho(\theta^{i-1}(x))$$

for $x \in N$. Put $\mathcal{N} := \pi(N)$. We use a conditional expectation $E_{\mathcal{N}}^{\mathcal{M}}$ defined by

$$E_{\mathcal{N}}^{\mathcal{M}}((e_{ij} \otimes x_{ij})_{ij}) = \frac{1}{2m+2} \pi \left(\sum_{i=-m}^{-1} \phi_\rho(\theta^{-i}(x_{ii})) + x_{00} + \sum_{i=1}^{m+1} \phi_\rho(\theta^{-i+1}(x_{ii})) \right).$$

We set $\tilde{\varphi} := \varphi\pi^{-1}E_N^{\mathcal{M}}$, and we have

$$\tilde{\varphi}((e_{ij} \otimes x_{ij})_{ij}) = \frac{1}{2m+2} \varphi \left(\sum_{i=-m}^{-1} \phi_{\rho}(\theta^{-i}(x_{ii})) + x_{00} + \sum_{i=1}^{m+1} \phi_{\rho}(\theta^{-i+1}(x_{ii})) \right).$$

Since $(\rho, \theta^n)_N = 0$ for any $n \in \mathbb{Z}$ by assumption, we have

$$\mathcal{N}' \cap \mathcal{M} = \sum_{i=-m}^{-1} \mathbb{C}e_{ii} \otimes (\rho\theta^i, \rho\theta^i)_N + \mathbb{C}e_{00} \otimes (\text{id}, \text{id})_N + \sum_{i=1}^{m+1} \mathbb{C}e_{ii} \otimes (\rho\theta^{i-1}, \rho\theta^{i-1})_N.$$

Hence $\{e_{ii}\}_{i=-m}^{m+1} \subset \mathcal{N} \vee (\mathcal{N}' \cap \mathcal{M}) \subset (\{e_{ii}\}_{i=-m}^{m+1})' \cap \mathcal{M}$. In particular, e_{ij} , $i \neq j$ is orthogonal to $\mathcal{N} \vee (\mathcal{N}' \cap \mathcal{M})$.

Let $\varepsilon > 0$. By applying Lemma 4.28 to $\mathcal{N} \subset \mathcal{M}$, we obtain a partition of unity $(q_r)_{r=1}^t$ of N_{ω} such that for all i with $i \neq 0$,

$$\left\| \sum_{r=1}^t \pi^{\omega}(q_r)(e_{i0} \otimes 1) \pi^{\omega}(q_r) \right\|_{\tilde{\varphi}^{\omega}} < \frac{\varepsilon}{2m+1} \|e_{i0} \otimes 1\|_{\tilde{\varphi}^{\omega}},$$

where $\pi^{\omega}: N^{\omega} \rightarrow \mathcal{M}^{\omega}$ is the natural extension of $\pi: N \rightarrow \mathcal{M}$. By direct computation, we obtain

$$\left\| \sum_{r=1}^t \pi^{\omega}(q_r)(e_{i0} \otimes 1) \pi^{\omega}(q_r) \right\|_{\tilde{\varphi}^{\omega}}^2 = \begin{cases} \frac{1}{2m+2} \sum_{r=1}^t \|\rho^{\omega}(\theta_{\omega}^i(q_r))q_r\|_{\varphi^{\omega}}^2 & \text{if } -m \leq i \leq -1, \\ \frac{1}{2m+2} \sum_{r=1}^t \|\rho^{\omega}(\theta_{\omega}^{i-1}(q_r))q_r\|_{\varphi^{\omega}}^2 & \text{if } 1 \leq i \leq m+1. \end{cases}$$

Since $\rho^{\omega}(\theta_{\omega}^{i-1}(q_r)), q_r \in (N^{\omega})_{\varphi^{\omega}}$, we have $\|\rho^{\omega}(\theta_{\omega}^i(q_r))q_r\|_{\varphi^{\omega}} = \|q_r \rho^{\omega}(\theta_{\omega}^i(q_r))\|_{\varphi^{\omega}}$. Hence we have

$$\sum_{r=1}^t \|q_r \rho^{\omega}(\theta_{\omega}^i(q_r))\|_{\varphi^{\omega}}^2 < \frac{\varepsilon^2}{(2m+1)^2} \quad \text{for all } |i| \leq m.$$

Summing up with i , we obtain

$$\sum_{r=1}^t \left(\sum_{|i| \leq m} \|q_r \rho^{\omega}(\theta_{\omega}^i(q_r))\|_{\varphi^{\omega}}^2 \right) < \frac{\varepsilon^2}{2m+1} \sum_{r=1}^t \|q_r\|_{\varphi^{\omega}}^2. \quad (4.1)$$

We set an index subset in \mathbb{N}

$$I_0 := \left\{ r \in \mathbb{Z} \mid 1 \leq r \leq t, \sum_{|i| \leq m} \|q_r \rho^{\omega}(\theta_{\omega}^i(q_r))\|_{\varphi^{\omega}} < \frac{\varepsilon}{(2m+1)^{1/2}} \|q_r\|_{\varphi^{\omega}} \right\}.$$

We set $q_0 := 1 - \sum_{r \in I_0} q_r$. We check that the family $\{q_0\} \cup \{q_r\}_{r \in I_0}$ is a desired one. On the size of q_0 , we have

$$\begin{aligned}
|q_0|_{\varphi^\omega} &= \sum_{r \in I \setminus I_0} |q_r|_{\varphi^\omega} < \sum_{r \in I \setminus I_0} \frac{(2m+1)^{1/2}}{\varepsilon} \sum_{|i| \leq m} \|q_r \rho^\omega(\theta_\omega^i(q_r))\|_{\varphi^\omega} \\
&\leq \frac{(2m+1)^{1/2}}{\varepsilon} \sum_{r=1}^t \sum_{|i| \leq m} \|q_r \rho^\omega(\theta_\omega^i(q_r))\|_{\varphi^\omega} \\
&< \frac{(2m+1)^{1/2}}{\varepsilon} \frac{\varepsilon^2}{2m+1} \sum_{r=1}^t \|q_r\|_{\varphi^\omega}^2 \quad (\text{by (4.1)}) \\
&= \frac{\varepsilon}{(2m+1)^{1/2}} < \varepsilon.
\end{aligned}$$

Note that N_ω and $\rho^\omega(N_\omega)$ are contained in $(N^\omega)_{\varphi^\omega}$. For any $x, y \in (N^\omega)_{\varphi^\omega}$, the inequality $|xy|_{\varphi^\omega} \leq \|x\|_{\varphi^\omega} \|y\|_{\varphi^\omega}$ holds. Using this inequality, we have, for $r \in I_0$,

$$\begin{aligned}
\sum_{|i| \leq m} |q_r \rho^\omega(\theta_\omega^i(q_r))|_{\varphi^\omega} &\leq \sum_{|i| \leq m} \|q_r\|_{\varphi^\omega} \|q_r \rho^\omega(\theta_\omega^i(q_r))\|_{\varphi^\omega} \\
&\leq \left(\sum_{|i| \leq m} \|q_r\|_{\varphi^\omega}^2 \right)^{1/2} \left(\sum_{|i| \leq m} \|q_r \rho^\omega(\theta_\omega^i(q_r))\|_{\varphi^\omega}^2 \right)^{1/2} \\
&< (2m+1)^{1/2} \|q_r\|_{\varphi^\omega} \cdot \frac{\varepsilon}{(2m+1)^{1/2}} \|q_r\|_{\varphi^\omega} \\
&= \varepsilon |q_r|_{\varphi^\omega}.
\end{aligned}$$

□

For an endomorphism trivially acting on $N_\omega^{\theta_\omega}$, we have the following.

Lemma 4.30. *Assume that $\rho \in \mathcal{E}_\theta$ acts on $N_\omega^{\theta_\omega}$ trivially. Then for any $\varepsilon > 0$ and any faithful state $\varphi \in N_*$, there exists $\delta > 0$ and a finite set of states $\mathcal{S} \subset N_*$ such that if $u \in U(N)$ satisfies $\|[u, \chi]\| < \delta$ for all $\chi \in \mathcal{S}$ and $|\theta(u) - u|_\varphi < \delta$, then we have $\|\rho(u) - u\|_\varphi < \varepsilon$.*

Proof. Take an increasing sequence of finite subsets $\{\mathcal{F}_\nu\}_{\nu=1}^\infty$ in N_* so that $\cup_{\nu=1}^\infty \mathcal{F}_\nu \subset N_*$ is dense. Suppose that the statement is false. Then there exist $\varepsilon_0 > 0$ and a faithful state $\varphi \in N_*$ such that for any $\nu \in \mathbb{N}$, we can take $u^\nu \in U(N)$ such that $\|[u^\nu, \chi]\| < 1/\nu$ for all $\chi \in \mathcal{F}_\nu$, $|\theta(u^\nu) - u^\nu|_\varphi < 1/\nu$ and $\|\rho(u^\nu) - u^\nu\|_\varphi \geq \varepsilon_0$ hold. Then the sequence $(u^\nu)_\nu$ is a centralizing sequence in N , and set $u := \pi_\omega((u^\nu)_\nu) \in N_\omega$. Since we have $|\theta(u^\nu) - u^\nu|_\varphi \rightarrow 0$ as $\nu \rightarrow \infty$ and u is centralizing, we easily see that $\theta(u^\nu) - u^\nu \rightarrow 0$ strongly* as $\nu \rightarrow \infty$. Hence $\theta_\omega(u) = u$. However we have $\|\rho^\omega(u) - u\|_{\varphi^\omega} \geq \varepsilon_0$, and this shows that ρ^ω is not trivial on $N_\omega^{\theta_\omega}$. This is a contradiction. □

The following lemma is directly proved from the previous lemma.

Lemma 4.31. *Assume that $\rho \in \mathcal{E}_\theta$ acts on $N_\omega^{\theta_\omega}$ trivially. Then for any $\varepsilon > 0$ and any faithful state $\varphi \in N_*$, there exists $\delta > 0$ such that if $u \in U(N_\omega)$ satisfies $|\theta_\omega(u) - u|_{\varphi^\omega} < \delta$, then $\|\rho^\omega(u) - u\|_{\varphi^\omega} < \varepsilon$.*

The following lemma plays an important role for our work in type III₀ case.

Lemma 4.32. *If $\rho \in \mathcal{E}_\theta$ acts on $N_\omega^{\theta_\omega}$ trivially, then $(\rho, \theta^n)_N \neq 0$ for some $n \in \mathbb{Z}$.*

Proof. Assume that $(\rho, \theta^n)_N = 0$ for all $n \in \mathbb{Z}$. We will derive a contradiction.

Step I. We prepare $\delta, \varepsilon, \varepsilon_1 > 0$, $m \in \mathbb{N}$ and the states $\psi, \tilde{\psi}, \varphi$ and $\tilde{\varphi}$ on N .

Take $0 < \varepsilon < 1$ and a faithful state $\psi \in N_*$. Set $\varphi := \psi \circ \phi_\rho$. For ε and the state φ , we can take $\delta > 0$ as in Lemma 4.31. Take $m \in \mathbb{N}$ enough large to satisfy $\sqrt{3/m} < \delta/2$ and $1/2 < 1 - 1/m - \sqrt{3/m} < 1$. Take $\varepsilon_1 > 0$ such that

$$1/2 < 1 - 1/m - \sqrt{3/m} - 2m\varepsilon_1 < 1. \quad (4.2)$$

Define the following faithful states on N

$$\tilde{\psi} := \frac{1}{m} \sum_{j=0}^{m-1} \psi \theta^j, \quad \tilde{\varphi} := \tilde{\psi} \circ \phi_\rho.$$

Since θ and ϕ_ρ commute, we trivially have

$$\tilde{\varphi} = \frac{1}{m} \sum_{j=0}^{m-1} \varphi \theta^j.$$

Note that N_ω and $\rho^\omega(N_\omega)$ are contained in $(N^\omega)_{\tilde{\varphi}^\omega}$.

Step II. We take a ‘‘Rohlin partition’’ $(q_i)_{i=0}^t$ in N_ω for $\{\rho \theta_\omega^n\}_{n=-m}^m \subset \text{End}(N^\omega)$.

For ε_1 , m and $\tilde{\varphi}$, we apply Lemma 4.29. Then there exists $t \in \mathbb{N}$ and a partition of unity $(q_i)_{i=0}^t$ in N_ω such that

$$\sum_{|n| \leq m} |q_i \rho^\omega(\theta_\omega^n(q_i))|_{\tilde{\varphi}^\omega} < \varepsilon_1 |q_i|_{\tilde{\varphi}^\omega} \quad \text{for all } 1 \leq i \leq t, \quad (4.3)$$

$$|q_0|_{\tilde{\varphi}^\omega} < \varepsilon_1. \quad (4.4)$$

Step III. We average each q_i and take \tilde{q}_i so that this is almost invariant under θ_ω and almost orthogonal to $\rho^\omega(\tilde{q}_i)$.

Take a Rohlin partition for $\{\mathcal{C}, \theta\}$ as [19, Lemma 10], i.e., a family of orthogonal projections $\{E_i\}_{i=0}^m$ in \mathcal{C} such that

- (1) $\theta(E_i) = E_{i+1}$ for $0 \leq i \leq m-1$,
- (2) $\left| \sum_{i=0}^{m-1} E_i \right|_\varphi \geq 1 - 1/m$,
- (3) $|E_0|_\varphi < 1/m$, $|E_m|_\varphi < 2/m$.

Next we average q_i along with the Rohlin partition as follows. For each $0 \leq i \leq t$, we define $\tilde{q}_i \in N_\omega$ by

$$\tilde{q}_i = \sum_{j=0}^{m-1} \theta_\omega^j(q_i) E_j.$$

It is clear that $\{\tilde{q}_i\}_{i=0}^t$ are orthogonal projections, and we have

$$\begin{aligned} \sum_{i=0}^t |\tilde{q}_i|_{\varphi^\omega} &= \sum_{i=0}^t \varphi^\omega(\tilde{q}_i) = \sum_{i=0}^t \sum_{j=0}^{m-1} \varphi^\omega(\theta_\omega^j(q_i) E_j) \\ &= \sum_{j=0}^{m-1} \varphi^\omega\left(\theta_\omega^j\left(\sum_{i=0}^t q_i\right) E_j\right) = \sum_{j=0}^{m-1} \varphi(E_j) \\ &> 1 - 1/m. \end{aligned} \tag{4.5}$$

For all $0 \leq i \leq t$, we have

$$\begin{aligned} \tilde{q}_i \rho^\omega(\tilde{q}_i) &= \sum_{j,k=0}^{m-1} \theta_\omega^j(q_i) E_j \rho^\omega(\theta_\omega^k(q_i) E_k) \\ &= \sum_{j,k=0}^{m-1} \theta_\omega^j(q_i) \rho^\omega(\theta_\omega^k(q_i)) E_j \rho^\omega(E_k) \\ &= \sum_{j,k=0}^{m-1} (\theta^\omega)^j(q_i \rho^\omega(\theta_\omega^{k-j}(q_i))) E_j \rho^\omega(E_k). \end{aligned}$$

Since N_ω and $\rho^\omega(N_\omega)$ are contained in $(N_\omega)_{\varphi^\omega}$, we have for $1 \leq i \leq t$,

$$\begin{aligned} |\tilde{q}_i \rho^\omega(\tilde{q}_i)|_{\varphi^\omega} &= \left| \sum_{j,k=0}^{m-1} (\theta^\omega)^j(q_i \rho^\omega(\theta_\omega^{k-j}(q_i))) E_j \rho^\omega(E_k) \right|_{\varphi^\omega} \\ &\leq \sum_{j,k=0}^{m-1} |(\theta^\omega)^j(q_i \rho^\omega(\theta_\omega^{k-j}(q_i))) E_j \rho^\omega(E_k)|_{\varphi^\omega} \\ &\leq \sum_{j,k=0}^{m-1} |(\theta^\omega)^j(q_i \rho^\omega(\theta_\omega^{k-j}(q_i)))|_{\varphi^\omega} \|E_j \rho^\omega(E_k)\| \\ &\leq \sum_{j=0}^{m-1} \sum_{|n| \leq m} |(\theta^\omega)^j(q_i \rho^\omega(\theta_\omega^n(q_i)))|_{\varphi^\omega} \\ &= \sum_{|n| \leq m} m |q_i \rho^\omega(\theta_\omega^n(q_i))|_{\tilde{\varphi}^\omega} \\ &< m \varepsilon_1 |q_i|_{\tilde{\varphi}^\omega} \quad (\text{by (4.3)}). \end{aligned} \tag{4.6}$$

By definition of \tilde{q}_i , we obtain $\theta_\omega(\tilde{q}_i) - \tilde{q}_i = -q_i E_0 + \theta_\omega^m(q_i) E_m$. Hence we have

$$\begin{aligned} \sum_{i=0}^t |\theta_\omega(\tilde{q}_i) - \tilde{q}_i|_{\varphi^\omega} &= \sum_{i=0}^t |-q_i E_0 + \theta_\omega^m(q_i) E_m|_{\varphi^\omega} = \sum_{i=0}^t (\varphi^\omega(q_i E_0) + \varphi^\omega(\theta_\omega^m(q_i) E_m)) \\ &= \varphi(E_0) + \varphi(E_m) \\ &< 1/m + 2/m = 3/m. \end{aligned}$$

By using (4.5), we have

$$\sum_{i=0}^t |\theta_\omega(\tilde{q}_i) - \tilde{q}_i|_{\varphi^\omega} < \frac{3}{m} \left(\frac{1}{m} + \sum_{i=0}^t |\tilde{q}_i|_{\varphi^\omega} \right).$$

Step IV. We show that most of $\{\tilde{q}_i\}_{i=0}^t$ are almost invariant under θ_ω .

We set an index set $I := \{i \in \mathbb{Z} \mid -1 \leq i \leq t\}$. We also put $X_{-1} := 0$, $Y_{-1} = 1/m$, $X_i := |\theta_\omega(\tilde{q}_i) - \tilde{q}_i|_{\varphi^\omega}$ and $Y_i := |\tilde{q}_i|_{\varphi^\omega}$ for $1 \leq i \leq t$. Then the above inequality yields

$$\sum_{i \in I} X_i < \frac{3}{m} \sum_{i \in I} Y_i.$$

We introduce the following index subsets of I :

$$I_0 := \{i \in I \mid X_i < \sqrt{3/m} Y_i\}, \quad I_1 := \{i \in I_0 \mid 1 \leq i \leq t\}.$$

By definition of I_1 , $\tilde{q}_i \neq 0$ if $i \in I_1$ and we have

$$|\theta_\omega(\tilde{q}_i) - \tilde{q}_i|_{\varphi^\omega} < \sqrt{3/m} |\tilde{q}_i|_{\varphi^\omega} < (\delta/2) |\tilde{q}_i|_{\varphi^\omega}. \quad (4.7)$$

We estimate of the total size of Y_i for $i \in I \setminus I_0$ as follows.

$$\begin{aligned} \sum_{i \in I \setminus I_0} Y_i &\leq \sum_{i \in I \setminus I_0} \sqrt{m/3} X_i \leq \sum_{i \in I} \sqrt{m/3} X_i \\ &< \sqrt{m/3} (3/m) \sum_{i \in I} Y_i = \sqrt{3/m} \sum_{i \in I} Y_i. \end{aligned}$$

This implies

$$\sum_{i \in I_0} Y_i = \sum_{i \in I} Y_i - \sum_{i \in I \setminus I_0} Y_i > (1 - \sqrt{3/m}) \sum_{i \in I} Y_i.$$

It is trivial that -1 is in I_0 . Hence we have

$$\begin{aligned}
\sum_{i \in I_0, 0 \leq i \leq t} |\tilde{q}_i|_\varphi &= \sum_{i \in I_0} Y_i - 1/m \\
&> (1 - \sqrt{3/m}) \sum_{i \in I} Y_i - 1/m \\
&= (1 - \sqrt{3/m}) (1/m + \sum_{i=1}^t Y_i) - 1/m \\
&= (1 - \sqrt{3/m}) \sum_{i=1}^t \varphi^\omega(\tilde{q}_i) - (1/m) \sqrt{3/m} \\
&> (1 - \sqrt{3/m}) (1 - 1/m) - (1/m) \sqrt{3/m} \quad (\text{by (4.5)}) \\
&= 1 - 1/m - \sqrt{3/m}. \tag{4.8}
\end{aligned}$$

Now we estimate the size of \tilde{q}_0 as follows.

$$\begin{aligned}
|\tilde{q}_0|_{\varphi^\omega} &= \sum_{j=0}^{m-1} \varphi^\omega(\theta^j(q_0) E_j) \leq \sum_{j=0}^{m-1} \varphi^\omega(\theta^j(q_0)) \\
&< \sum_{j=0}^{m-1} |q_0|_{(\varphi \theta^j)^\omega} = m |q_0|_{\tilde{\varphi}^\omega} \\
&< m \varepsilon_1. \quad (\text{by (4.4)})
\end{aligned}$$

Together with this inequality and (4.8), we have

$$\sum_{i \in I_1} |\tilde{q}_i|_{\varphi^\omega} > 1 - 1/m - \sqrt{3/m} - m \varepsilon_1. \tag{4.9}$$

Now we set $p_0 := 1 - \sum_{i \in I_1} \tilde{q}_i$. Then by (4.9),

$$|p_0|_{\varphi^\omega} < 1/m + \sqrt{3/m} + m \varepsilon_1, \tag{4.10}$$

and

$$\begin{aligned}
|\theta_\omega(p_0) - p_0|_{\varphi^\omega} &= \left| \sum_{i \in I_1} (\theta_\omega(\tilde{q}_i) - \tilde{q}_i) \right|_{\varphi^\omega} \leq \sum_{i \in I_1} |\theta_\omega(\tilde{q}_i) - \tilde{q}_i|_{\varphi^\omega} \\
&< \sum_{i \in I_1} (\delta/2) |\tilde{q}_i|_{\varphi^\omega} \quad (\text{by (4.7)}) \\
&\leq \delta/2. \tag{4.11}
\end{aligned}$$

Step V. We sum up $\{p_0\} \cup \{\tilde{q}_i\}_{i \in I_1}$ with phases and obtain unitaries almost invariant under θ_ω . We derive a contradiction by studying how ρ^ω acts on them.

For $z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, we define the unitary element

$$u(z) = p_0 + \sum_{i \in I_1} z^i \tilde{q}_i.$$

Then we have for all $z \in \mathbb{T}$,

$$\begin{aligned} |\theta_\omega(u(z)) - u(z)|_{\varphi^\omega} &\leq |\theta_\omega(p_0) - p_0|_{\varphi^\omega} + \sum_{i \in I_1} |\theta_\omega(\tilde{q}_i) - \tilde{q}_i| \\ &< \delta/2 + (\delta/2) \sum_{i \in I_1} |\tilde{q}_i|_{\varphi^\omega} \leq \delta. \quad (\text{by (4.7) and (4.11)}) \end{aligned}$$

Since we have taken ε and δ as in Lemma 4.31, we have

$$\|\rho^\omega(u(z)) - u(z)\|_{\varphi^\omega} < \varepsilon \quad \text{for all } z \in \mathbb{T}.$$

Making use of $\int_{\mathbb{T}} z^k dz = 0$ if $k \neq 0$, we have

$$\begin{aligned} \varepsilon^2 &\geq \int_{\mathbb{T}} \|\rho^\omega(u(z)) - u(z)\|_{\varphi^\omega}^2 dz \\ &= \int_{\mathbb{T}} (2 - \varphi^\omega(u(z)^* \rho^\omega(u(z))) - \varphi^\omega(\rho^\omega(u(z)^*) u(z))) dz \\ &= 2 - 2 \left(\varphi^\omega(p_0 \rho^\omega(p_0)) + \sum_{i \in I_1} \varphi^\omega(\tilde{q}_i \rho^\omega(\tilde{q}_i)) \right) \\ &> 2 - 2 \left(\varphi^\omega(p_0) + \sum_{i \in I_1} m \varepsilon_1 |q_i|_{\tilde{\varphi}^\omega} \right) \quad (\text{by (4.6)}) \\ &> 2 - 2 \left(1/m + \sqrt{3/m} + m \varepsilon_1 + m \varepsilon_1 \right) > 1 > \varepsilon^2 \quad (\text{by (4.2) and (4.10)}). \end{aligned}$$

However this is a contradiction. \square

The following lemma is similar to [17, Proposition 3.4 (1)], which is stated about a canonical extension.

Lemma 4.33. *Let $\rho \in \mathcal{E}_\theta$ such that $(\rho(N)' \cap N)^\theta = \mathbb{C}$. If there exists $n \in \mathbb{N}$ such that $(\rho, \theta^n)_N \neq 0$, then there exists a Hilbert space \mathcal{H} in N such that $\rho = \rho_{\mathcal{H}} \circ \theta^n$.*

Proof. We may and do assume that $(\rho, \text{id})_N \neq 0$ by considering $\rho \theta^{-n}$ for ρ in case of $(\rho, \theta^n)_N \neq 0$. Note the fact that $(\text{id}, \rho)_N$ is $\rho(N)' \cap N$ - $Z(N)$ -bimodule, that is, if $a \in \rho(N)' \cap N$, $X \in (\text{id}, \rho)_N$ and $b \in Z(N)$, then $aXb \in (\text{id}, \rho)_N$. Also note that $(\text{id}, \rho)_N$ is globally invariant under θ because ρ and θ commute.

Step I. We show that for any non-zero projection $p \in \rho(N)' \cap N$, there exists a non-zero partial isometry $V \in (\text{id}, \rho)_N$ such that $pV \neq 0$.

Assume that such an element does not exist. Then for any $X \in (\text{id}, \rho)_N$, we have $pX = 0$, and $\theta^n(p)\theta^n(X) = 0$ for all $n \in \mathbb{Z}$. Since $\theta^n((\text{id}, \rho)_N) = (\text{id}, \rho)_N$, we have $\theta^n(p)(\text{id}, \rho)_N = 0$ for all $n \in \mathbb{Z}$. This implies that $\bigvee_{n \in \mathbb{Z}} \theta^n(p)(\text{id}, \rho)_N = 0$. However the projection $\bigvee_{n \in \mathbb{Z}} \theta^n(p)$ is contained in $(\rho(N)' \cap N)^\theta = \mathbb{C}$, and it is equal to 1. This shows $(\text{id}, \rho)_N = 0$. This is a contradiction. Hence there exists a non-zero $X \in (\text{id}, \rho)_N$ such that $pX \neq 0$. Let $X = V|X|$ be the polar decomposition of X . Then it is easy to see that $V \in (\text{id}, \rho)_N$. The partial isometry V satisfies $pV \neq 0$.

Step II. We show the following: Let $p \in \rho(N)' \cap N$ be a non-zero projection. Then there exists a partial isometry $V \in (\text{id}, \rho)_N$ such that $VV^* \leq p$ and $V^*V = z_N(p)$, where $z_N(p)$ is the central support projection of p in N .

For partial isometries $V, W \in p(\text{id}, \rho)_N$, we define the relation $V \prec W$ by $V = WV^*V$, and this gives an inductive order on the set $p(\text{id}, \rho)_N$ as in the proof of [17, Proposition 3.4]. Take W a partial isometry from $p(\text{id}, \rho)_N$ which is maximal with respect to this order. Note that $WW^* \leq p$, and $W^*W \leq z_N(p)$. Assume that $W^*W \neq z_N(p)$. We set a central projection $z_0 := z_N(p) - W^*W$. Using Step I for a non-zero projection $pz_0 \in \rho(N)' \cap N$, we obtain a non-zero partial isometry $W_0 \in (\text{id}, \rho)_N$ such that $W_0 = pz_0W_0$. Then $W_0 \in p(\text{id}, \rho)_N$ and $W_0^*W_0 \leq z_0$. Hence $W_0^*W_0W^*W = 0$. Using $W^*W \in Z(N)$, we have

$$W^*W_0 = (W^*W)W^*W_0 = W^*W_0(W^*W) = W^*W_0z_0(W^*W) = 0.$$

Hence $W + W_0 \in p(\text{id}, \rho)_N$ is a partial isometry such that $W \prec W + W_0$. This is a contradiction. Therefore $W^*W = z_N(p)$.

Step III. Take any non-zero projection z in $Z(N)$. We show that there exists a non-zero projection $z_1 \leq z$ in $Z(N)$ and a family of partial isometries $(V_i)_{i \in I}$ in $(\text{id}, \rho)_N$ such that

$$V_i^*V_i = z_1 \text{ for all } i \in I \text{ and } \sum_{i \in I} V_iV_i^* = z_1.$$

Using Step II for the projection z , we can take a family of partial isometries $(W_i)_{i \in I}$ in $(\text{id}, \rho)_N$ whose range projections are maximally orthogonal and $W_i^*W_i = z$ for all $i \in I$. We set $p := z - \sum_{i \in I} W_iW_i^*$. Again by Step II, we can take a partial isometry $W \in (\text{id}, \rho)_N$ such that $pW = W$ and $W^*W = z_N(p)$. If $z_N(p) = z$, then a extended family $(W_i)_{i \in I} \cup \{W\}$ contradicts with the maximality. Hence $z_N(p) \neq z$, and we set $z_1 := z - z_N(p) \neq 0$ and $V_i := z_1W_i$. It is trivial that $(V_i)_{i \in I}$ are contained in $(\text{id}, \rho)_N$ and satisfy $V_i^*V_j = \delta_{i,j}V_i^*V_i$ for all $i, j \in I$. Since $z_1p = 0$, we have $z_1 = \sum_{i \in I} V_iV_i^*$.

Step IV. We show that there exists a partition of unity $(z_\lambda)_{\lambda \in \Lambda}$ in $Z(N)$ and a family of partial isometries $(V_i^\lambda)_{i \in I_\lambda}$ in $(\text{id}, \rho)_N$ for each $\lambda \in \Lambda$ such that

$$(V_i^\lambda)^*V_i^\lambda = z_\lambda \text{ for all } i \in I_\lambda \text{ and } \sum_{i \in I_\lambda} V_i^\lambda(V_i^\lambda)^* = z_\lambda.$$

Let $(z_\lambda)_{\lambda \in \Lambda} \subset Z(N)$ be a maximal family of orthogonal projections which possesses such families of partial isometries. Assume that $\sum_{\lambda \in \Lambda} z_\lambda \neq 1$. By Step III, there exist a non-zero projection $z_1 \in Z(N)$ and a family of partial isometries $(V_i)_{i \in I}$ in $(\text{id}, \rho)_N$ such that $z_1z_\lambda = 0$ for all $\lambda \in \Lambda$ and

$$V_i^*V_i = z_1 \text{ for all } i \in I \text{ and } \sum_{i \in I} V_iV_i^* = z_1.$$

The family $(z_\lambda)_{\lambda \in \Lambda} \cup \{z_1\}$ contradicts with the maximality. Hence $\sum_{\lambda \in \Lambda} z_\lambda = 1$.

Step V. Let z_λ and V_i^λ as in Step IV. We show that the cardinality of each I_λ is equal to each other.

Take $\lambda_1, \lambda_2 \in \Lambda$. By ergodicity of $\{\mathcal{C}, \theta\}$, there exists $n \in \mathbb{Z}$ such that $z_{12} := \theta^n(z_{\lambda_1})z_{\lambda_2} \neq 0$. We set $W_i^{\lambda_j} := (\theta^n)^{2-j}(V_i^{\lambda_j})z_{12}$ for $i \in I_{\lambda_j}$ and $j = 1, 2$. Trivially $W_i^{\lambda_j}$ is contained in $(\text{id}, \rho)_N$. Then we have, for each $j = 1, 2$,

$$(W_i^{\lambda_j})^* W_i^{\lambda_j} = z_{12} \text{ for all } i \in I_{\lambda_j} \text{ and } \sum_{i \in I_{\lambda_j}} W_i^{\lambda_j} (W_i^{\lambda_j})^* = z_{12}.$$

Note that $(W_i^{\lambda_1})^* W_j^{\lambda_2} \in Z(N)$ for $i \in I_1$ and $j \in I_2$. Let $|I_\lambda| \leq \infty$ be the cardinality of I_λ for $\lambda \in \Lambda$. Then

$$\begin{aligned} |I_{\lambda_1}| z_{12} &= \sum_{i \in I_{\lambda_1}} (W_i^{\lambda_1})^* z_{12} W_i^{\lambda_1} = \sum_{i \in I_{\lambda_1}} \sum_{j \in I_{\lambda_2}} (W_i^{\lambda_1})^* W_j^{\lambda_2} (W_j^{\lambda_2})^* W_i^{\lambda_1} \\ &= \sum_{j \in I_{\lambda_2}} \sum_{i \in I_{\lambda_1}} (W_j^{\lambda_2})^* W_i^{\lambda_1} (W_i^{\lambda_1})^* W_j^{\lambda_2} = \sum_{j \in I_{\lambda_2}} (W_j^{\lambda_2})^* z_{12} W_j^{\lambda_2} \\ &= \sum_{j \in I_{\lambda_2}} z_{12} = |I_{\lambda_2}| z_{12}. \end{aligned}$$

Hence $|I_{\lambda_1}| = |I_{\lambda_2}|$.

Step VI. By Step V, we may and do assume that each index set I_λ is the same index set I . We set $V_i := \sum_{\lambda \in \Lambda} V_i^\lambda$. It is trivial that V_i is an isometry in $(\text{id}, \rho)_N$ and $\sum_{i \in I} V_i V_i^* = 1$. Then the Hilbert space spanned by $(V_i)_{i \in I}$ implements ρ . \square

• *Proof of Theorem 4.12 for hyperfinite type III_0 factors.*

It suffices to show that $\text{Cnd}(M)_{\text{irr}} \subset \text{End}(M)_{\text{m}}$ as before. Let $M = N \rtimes_\theta \mathbb{Z}$ be the discrete decomposition with the implementing unitary U . Take $\sigma \in \text{Cnd}(M)_{\text{irr}}$. By Lemma 4.25, we may and do assume that σ and $\hat{\theta}$ commute and moreover $\sigma(U) = U$. We set $\rho := \sigma|_N$. Then the restriction $\phi_\sigma|_N$ is a left inverse of ρ and it commutes with θ . Hence $\rho \in \mathcal{E}_\theta$. Since M_ω is naturally identified with $(N_\omega)^{\theta_\omega}$ [19, Lemma 7], $\sigma^\omega = \text{id}$ on $(N_\omega)^{\theta_\omega}$. Then by Lemma 4.32, $(\sigma, \theta^n)_N \neq 0$ for some $n \in \mathbb{Z}$. By Lemma 4.33, we can find a Hilbert space \mathcal{H} in N such that $\sigma = \rho_{\mathcal{H}} \theta^n$. Thus ρ is a modular endomorphism by Lemma 4.26. \square

Remark 4.34. In this paper, we have studied centrally trivial endomorphisms with finite indices. However, as we claimed before, central triviality has the meaning even for endomorphisms with infinite indices if they have left inverses. Hence it is natural to consider the generalization of Theorem 4.12. It seems that Lemma 4.32, 4.33 play important roles because in them we have not assumed that ρ has finite index, but that study is beyond our scopes at the present.

Acknowledgments. The authors would like to thank Yasuyuki Kawahigashi and Masaki Izumi for encouragement on this work. They are also grateful to Robert Longo and Sebastiano Carpi for various useful comments and discussions. A part of this work was done while the first named author stayed at Fields Institute and the second named author stayed at Università di Roma “Tor Vergata”, and completed when the second named author stayed at Katholieke Universiteit Leuven. They express gratitude for their warm hospitality.

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